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GENERAL RECURSIVE FUNCTIONS

JULIA ROBINSON

1. Introduction. A primitive recursive function is one which can be obtained from the initial functions I_{nk} $(1 \leqq k \leqq n)$, O_n $(n \geqq 0)$, and S, by repeated substitution and recursion. Here

$$I_{nk}(x_1, \cdots, x_n) = x_k, \qquad O_n(x_1, \cdots, x_n) = 0,$$

S denotes the successor function, and the recursion scheme has the form

$$F(\mathfrak{x}, 0) = A\mathfrak{x}, \qquad F(\mathfrak{x}, Sy) = B(\mathfrak{x}, y, F(\mathfrak{x}, y)),$$

where we have put $\mathfrak{x} = (x_1, \cdots, x_n)$, $n \geqq 0$.[1]

The class of general recursive functions is obtained if we allow one additional scheme for defining functions, namely

$$F\mathfrak{x} = \mu y\{A(\mathfrak{x}, y) = 0\},$$

where the symbol on the right denotes the smallest y such that $A(\mathfrak{x}, y) = 0$, under the assumption that there is such a y for each $\mathfrak{x}$.

Kleene showed that this definition of general recursive function is equivalent to Herbrand-Gödel metamathematical definition.[2] In this paper we shall be concerned with the mathematical (as opposed to metamathematical) aspects of the theory of general recursive functions. Starting from the definition stated, we shall investigate the possible restrictions on the defining schemes. Part of the results obtained are already known from the work of Kleene, but we go further in this direction than Kleene did. No previous knowledge of general recursive functions is assumed in this paper.

It will be convenient to have a logical symbolism to express the conditions that appear in applications of the μ-rule. We shall use: $\bigwedge$ (for every), $\bigvee$ (there exists), $\wedge$ (and), $\vee$ (or), $\sim$ (not), $\rightarrow$ (if — then), $\leftrightarrow$ (if and only if).

The following equivalences will be useful:

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[1] Further discussion of this definition may be found, for example, in R. M. Robinson, *Primitive recursive functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 925–942. This paper will be referred to as PRF. The notation of the present paper is the same as in PRF.

[2] See S. C. Kleene, *General recursive functions of natural numbers*, Math. Ann. vol. 112 (1936) pp. 727–766 and *Recursive predicates and quantifiers*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 41–73.

$$a = 0 \wedge b = 0 \leftrightarrow a + b = 0, \qquad a = 0 \vee b = 0 \leftrightarrow ab = 0,$$
$$a \neq 0 \leftrightarrow 0^a = 0.$$

In connection with the last formula, we recall that $0^0 = 1$. Notice also that

$$a = b \leftrightarrow (a - b) + (b - a) = 0,$$
$$a \geqq b \leftrightarrow [(a - b) + b] - a = 0$$

where an arbitrary difference function $x - y$ is used; that is, $x - y$ has the usual meaning when $x \geqq y$, but a completely arbitrary meaning for $x < y$.

At various places, we shall need pairing functions, that is functions J, K, and L such that

$$KJ(x, y) = x \quad \text{and} \quad LJ(x, y) = y.$$

It will also be convenient to have a one-to-one correspondence between pairs of numbers (x, y) and numbers z. This is expressed by the identity

$$J(Kz, Lz) = z.$$

Furthermore, sometimes we shall want $J(x, y)$ to be a monotone increasing function of x and y. All of these conditions are satisfied by Cantor's diagonal enumeration. This one-to-one correspondence between pairs of numbers (x, y) and numbers z is established by the equation

$$2z = (x + y)^2 + 3x + y.$$

Notice that

$$8z + 1 = (2x + 2y + 1)^2 + 8x$$

and hence $[(8z+1)^{1/2}]$ is either $2x+2y+1$ or $2x+2y+2$. It follows easily that J, K, L can all be obtained by substitution alone, if we adjoin

$$x^2, \quad x + y, \quad x - y, \quad [x/2], \quad [x^{1/2}]$$

to the initial functions.

2. **Various forms of the μ-rule.** A more general form of the μ-rule is

$$F\mathfrak{x} = \mu y\{A_1(\mathfrak{x}, y) = A_2(\mathfrak{x}, y)\}.$$

This can be reduced to the standard form

$$F\mathfrak{x} = \mu y\{A(\mathfrak{x}, y) = 0\}$$

by putting $A(\mathfrak{x}, y) = |A_1(\mathfrak{x}, y) - A_2(\mathfrak{x}, y)|$. Consider next the more special form

$$F\mathfrak{x} = \iota y\{B(\mathfrak{x}, y) = 0\}$$

where the right side denotes the only y such that $B(\mathfrak{x}, y) = 0$, under the assumption that there is just one such y for each $\mathfrak{x}$. This is readily seen to be equivalent to the preceding form if we take

$$B(\mathfrak{x}, y) = A(\mathfrak{x}, y) + \sum_{z<y} 0^{A(\mathfrak{x}, z)}.$$

Indeed, $B(\mathfrak{x}, y) = 0$ is equivalent to $A(\mathfrak{x}, y) = 0$ and $A(\mathfrak{x}, z) \neq 0$ for $z < y$. Notice that $B(\mathfrak{x}, y)$ is primitive recursive if $A(\mathfrak{x}, y)$ is.

By successive applications of pairing functions, we can reduce the number of components of $\mathfrak{x} = (x_1, \cdots, x_n)$ which occur in applications of the μ-rule to one. Consider for example the reduction from two components to one. We are given

$$F(x_1, x_2) = \mu y\{A(x_1, x_2, y) = 0\}.$$

Putting

$$B(x, y) = A(Kx, Lx, y)$$

and

$$Gx = \mu y\{B(x, y) = 0\},$$

we see that

$$F(x_1, x_2) = GJ(x_1, x_2).$$

The same reduction is also valid with ι in place of μ.

We shall next discuss the relation between the following two forms of the μ-rule:

$$Fx = \mu y\{A(x, y) = 0\},$$

$$Gx = \mu z\{Bz = x\}.$$

It is easily seen that

$$Fx = L\mu z\{A(Kz, Lz) = 0 \wedge Kz = x\},$$

where J, K, and L are Cantor's pairing functions. Indeed, the equation $z = J(x, y)$, with x fixed, establishes a one-to-one correspondence between values of y satisfying $A(x, y) = 0$ and values of z such that $A(Kz, Lz) = 0 \wedge Kz = x$. Also, the smallest y corresponds to the smallest z. It follows that

$$L\mu z\{0^{A(Kz,Lz)}Kz = x\} = \begin{cases} Fx & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Thus, putting

$$Bz = 0^{A(Kz,Lz)}Kz$$

and defining Gx as above, we see that

$$Fx = LGx + 0^{x}F0.$$

We may also put $Fx = HGx$, where

$$Hy = \begin{cases} Ly & \text{if } y \neq G0, \\ F0 & \text{if } y = G0. \end{cases}$$

(H is a primitive recursive function.)

The situation is somewhat different for the ι-rule. A more elaborate method is needed for reducing

$$Fx = \iota y\{A(x, y) = 0\}$$

to the use of

$$Gx = \iota z\{Bz = x\}$$

and certain elementary substitutions. As before, we have

$$Fx = L\iota z\{A(Kz, Lz) = 0 \wedge Kz = x\}$$

or putting

$$Cz = \operatorname{sgn} A(Kz, Lz),$$

also

$$Fx = L\iota z\{Cz = 0 \wedge Kz = x\}.$$

For each x there is a unique z such that $Kz = x$ and $Cz = 0$. It is these values of z in which we are interested. We wish to construct a function B from which these values can be read in a simple way, but also so that $\iota z\{Bz = x\}$ is a legitimate operation, that is, so that for every x there is exactly one z with $Bz = x$, even though the value of z may be immaterial for our purposes.

We shall construct a function B such that

$$B(2z) = 2x \leftrightarrow Cz = 0 \wedge Kz = x.$$

The function B will assume all distinct odd values for odd arguments and for even arguments $2z$ for which $Cz = 1$. This will insure the legitimacy of the operation $\iota z\{Bz = x\}$. Such a function B may be de-

fined as follows:

$$B(2z) = \begin{cases} 2Kz & \text{if } Cz = 0, \\ 2z - 1 + 2\sum_{u \leq z} Cu & \text{if } Cz = 1, \end{cases}$$

$$B(2z+1) = 2z + 1 + 2\sum_{u \leq z} Cu.$$

Notice that B can be obtained from A by substitutions and primitive recursions, and hence is primitive recursive if A is. Finally, we see that if G is defined as above, we have

$$Fx = L[G(2x)/2].$$

REMARK. Thus, if substitution and primitive recursion are available, the strongest and weakest forms of the μ-rule are equivalent.

3. Elimination of recursion. In this section we shall show that all general recursive functions can be obtained without the use of the recursion scheme, that is, by using only substitution and the μ-rule, if we adjoin suitable functions to the initial functions. Indeed, we shall see that it is sufficient to adjoin $x+y$ and Q if we use the μ-rule in the form $F\xi = \mu y\{A(\xi, y) = 0\}$,[3] and to adjoin $x+y$ and E if we use the more special form of the μ-rule $Fx = \mu y\{Ay = x\}$. Here $Ex = x - [x^{1/2}]^2$ is the excess of x over a square, and $Qx = 0^{Bx}$ is the characteristic function of squares. In the latter case, in place of E we may use either K or L. I do not know whether the recursion scheme can be omitted if the ι-rule is substituted for the μ-rule and a finite number of primitive recursive functions are adjoined to the initial functions.

We first recall a device due to Gödel.[4] Any finite sequence of numbers $a_0, a_1, a_2, \cdots, a_y$ can be obtained as remainders when a fixed number is divided by $1+v(z+1)$, for $z = 0, 1, 2, \cdots, y$. Denoting by $R(u, t)$ the remainder when u is divided by t, the required conditions are

$$R(u, 1 + v(z+1)) = a_z \qquad \text{for } 0 \leq z \leq y.$$

These conditions are equivalent to

[3] A similar result was proved by Kleene, adjoining the functions $x+y$, xy, and $0^{|x-y|}$. The method of eliminating recursion is essentially the same as in the present paper. See Kleene, *A note on recursive functions*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 544–546.

[4] See Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme* I, Monatshefte für Mathematik und Physik vol. 38 (1931) pp. 173–198.

$$u \equiv a_z \pmod{1 + v(z+1)}, \qquad a_z < 1 + v(z+1).$$

We need only to choose v sufficiently large and so that the moduli are relatively prime, and then choose u satisfying the congruences.

If we put $w = J(u, v)$, then the sequence is determined from a single number w. Putting for brevity

$$T_z w = R(Kw, 1 + Lw \cdot Sz),$$

we have

$$a_z = T_z w \qquad \text{for } z = 0, 1, 2, \cdots y.$$

Thus $T_z w$ denotes the zth term in the sequence determined by w. (Here J can be taken to be any pairing function with suitably associated functions K and L.)

We are now ready to consider the elimination of recursive definitions. Suppose that F is defined recursively from A and B by the equations

$$F(\mathfrak{x}, 0) = A\mathfrak{x}, \qquad F(\mathfrak{x}, Sy) = B(\mathfrak{x}, y, F(\mathfrak{x}, y)).$$

The sequence of numbers

$$F(\mathfrak{x}, 0), F(\mathfrak{x}, 1), \cdots, F(\mathfrak{x}, y)$$

may be represented by a single number w in the manner described above. Choosing the smallest possible value for w, we see that

$$F(\mathfrak{x}, y) = T_y \mu w \{T_0 w = A\mathfrak{x} \wedge \bigwedge_z [z < y \to T_{Sz} w = B(\mathfrak{x}, z, T_z w)]\}.$$

This may also be written in the form

$$F(\mathfrak{x}, y) = T_y \mu w \{T_0 w = A\mathfrak{x} \wedge \mu z \{T_{Sz} w \neq B(\mathfrak{x}, z, T_z w) \vee z = y\} = y\}.$$

Thus we see that $F(x, y)$ has been obtained from the given functions A and B, and the function T by two applications of the μ-rule. These are not in standard form, but can be reduced to standard form by elementary transformations, making use of the functions $x+y$, $x-y$, xy, and 0^x. The function T is obtained by substitution from K, L, R, and S. Thus to make possible the elimination of recursion, it is sufficient to adjoin $x+y$, $x-y$, xy, 0^x, K, L, and R to the usual initial functions. All general recursive functions can then be obtained using substitution and the μ-rule in the standard form.

We shall show that actually it is sufficient to adjoin $x+y$ and Q to the initial functions.

We note first that $0^x = QSS(Qx+QSx)$. We shall next define a function $x^{1/2}$. This symbol, in distinction to $[x^{1/2}]$, will be used to de-

note any function which gives the square root of x when x is a square, but whose values for other arguments are immaterial. A suitable definition is

$$x^{1/2} = \mu y\{Q(x + 2y + 1) = 1\}.$$

The condition in braces may also be written in the standard form $0^{Q\delta(x+y+z)} = 0$.

It was proved by Euler that it is impossible to find four distinct squares forming an arithmetic progression. Using this fact, we may define x^2 as follows:

$$\begin{aligned} x^2 &= \mu y\{Qy = 1 \wedge Q(y + 3y^{1/2} + x + 4) \\ &= 1 \wedge Q(y + 2y^{1/2} + 2x + 4) \\ &= 1 \wedge Q(y + y^{1/2} + 3x + 4) = 1\}. \end{aligned}$$

In the first place, $y = x^2$ satisfies the condition. Conversely, if y satisfies the condition, then y is a square, so that $y^{1/2}$ indeed represents the square root of y. Thus $y+4y^{1/2}+4$ is also a square so that the condition requires four squares in arithmetic progression. They must all be equal, hence $y^{1/2} = x$ or $y = x^2$. Thus x^2 is the only y satisfying the condition in braces. The condition may be reduced to standard form by using

$$a \neq 0 \wedge b \neq 0 \wedge c \neq 0 \wedge d \neq 0 \leftrightarrow 0^a + 0^b + 0^c + 0^d = 0.$$

The following definitions may now be made:

$$\begin{aligned} x - y &= \mu z\{Q((x + y)^2 + x + 3y + 1 + z) = 1\}, \\ [x/2] &= \mu y\{Sy + Sy > x\}, \\ [x^{1/2}] &= \mu y\{(Sy)^2 > x\}, \\ x \cdot y &= [(((x + y)^2 - x^2) - y^2)/2], \\ [x/y] &= \mu z\{y \cdot Sz > x \vee y = 0\}, \\ R(x, y) &= x - y[x/y]. \end{aligned}$$

The conditions involving inequalities may be reduced to standard form using the difference function. Finally, K and L may also be obtained by substitution. Thus it is indeed sufficient to adjoin $x+y$ and Q to the initial functions in order to obtain all general recursive functions by substitution and use of the μ-rule.

We shall now prove that all general recursive functions can be obtained by substitution and inversion $A^{-1}x = \mu y\{Ay = x\}$ if we adjoin $x+y$ and E to the initial functions. The reduction of the general form of the μ-rule to this form required use of the functions J,

K, L, $x+y$, xy, and 0^x. We needed $x+y$ and Q in the general case. All of these functions can be obtained from

$$x + y, \quad x - y, \quad x^2, \quad [x/2], \quad [x^{1/2}], \quad 0^x$$

by subtitution. Thus we need only to show that these functions can be obtained from the usual initial functions and $x+y$ and E by substitution and inversion.

In the first place, we see that $E^{-1}(2x) = x^2 + 2x$. It follows easily that

$$E(E^{-1}(2x + 2y) + 3x + y + 4) = x - y \qquad \text{if } x \geqq y.$$

Now we define

$$x^2 = E^{-1}(2x) - 2x.$$

We may then put

$$\operatorname{sgn} x = ES(x^2), \qquad O^x = 1 - \operatorname{sgn} x.$$

It is easily seen that the predecessor function P can defined by

$$Px = E\mu y\{ESy = x\}.$$

Now observing that $ESSE^{-1}$ is the characteristic function of even numbers, and that E assumes all values for even arguments, we find that

$$[x/2] = E\mu y\{2Ey + ESSE^{-1}y = x\}.$$

Clearly,

$$x^{1/2} = [EPx/2] + \operatorname{sgn} x$$

if x is a square, and

$$[x^{1/2}] = (x - Ex)^{1/2}.$$

We have thus defined all the required functions.

Finally, we observe that instead of E we may equally well adjoin either K or L to the initial functions (the sum $x+y$ being adjoined in any case). The pattern of values assumed by K and L is indicated in the following table:

$$x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots,$$
$$Kx = 0, 0, 1, 0, 1, 2, 0, 1, 2, 3, \cdots,$$
$$Lx = 0, 1, 0, 2, 1, 0, 3, 2, 1, 0, \cdots.$$

In particular, Kx is the excess of x over a triangular number. We

notice that $K^{-1}x = x(x+3)/2$. The following formulas are then easily verified:

$$K(K^{-1}(x+y)+2x+3) = x-y \qquad \text{for } x \geqq y,$$

$$x^2 = 2K^{-1}x - 3x, \qquad Px = K\mu y\{KSy = x\},$$

$$Kx + Lx = KP(x - Kx) + KS(x - Kx).$$

Thus Lx can be obtained by subtraction. Noting that of all representations of x in the form u^2+v, the minimum value of $u+v$ is attained only when $u = [x^{1/2}]$ and $v = Ex$, we see that $J(u, v)$ is smallest in this case. Hence

$$Ex = L\mu y\{(Ky)^2 + Ly = x\}.$$

Thus K may be used as an initial function in place of E. To see that L may equally well be used, we need only note the equation

$$Kx = L(x + 2Lx + 2).$$

4. **General recursive functions of one variable.** The results of this section, as far as Theorem 3, are exactly parallel to those of PRF, §7. The omitted proofs can be copied from there practically verbatim.

It is easy to see that all general recursive functions of more than one variable can be obtained from general recursive functions of one variable, the function $x+y$, and the various identity functions, by substitution alone. Furthermore, as we shall show, all general recursive functions of one variable can be obtained without using functions of more variables in the process.

THEOREM 1. *The functions of one variable which are obtainable from certain initial functions by substitution and inversion are also obtainable if substitution is not allowed except for defining functions of one variable.*

THEOREM 2. *If the initial functions include the identity and zero functions, and the function $x+y$, but no other function of more than one variable, then all functions of one variable obtainable by substitution and inversion are also obtainable by repeated use of the three formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^{-1}x$$

to define F when A and B are known functions of one variable. The third formula is used only when B assumes all values.

THEOREM 3. *All general recursive functions of one variable can be obtained by starting with the two functions S and E, and repeatedly using any of the formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^{-1}x$$

to construct a new function from the known functions A and B. The third formula is used only when B assumes all values.

PROOF. The identity and zero functions of one variable, $I = I_{11}$ and $O = O_1$, can be defined by

$$Ix = EE^{-1}x, \qquad Ox = ESE^{-1}(x + x).$$

Thus by Theorem 2, we can obtain all the functions of one variable which we could get by substitution and inversion from all the identity and zero functions, $x+y$, S, and E. But by §3, all general recursive functions can be so obtained.

REMARK. In place of E in this theorem, we may also use either K or L. In virtue of the results of §3, we need only note the definitions of I and O,

$$Ix = KK^{-1}x, \qquad Ox = KSK^{-1}x,$$
$$Ix = LL^{-1}x, \qquad Ox = L(Ix + L^{-1}x).$$

Furthermore, in place of S we may also use SO. For without using S we can define I, and then we put $Sx = Ix + SOx$.

REMARK. Comparing this theorem with the corresponding theorem of PRF, §7, we see that the only difference is the substitution of the scheme $Fx = B^{-1}x$ for the scheme $Fx = B^xO$, the result of this change being that we obtain all general recursive functions instead of just primitive recursive functions.

We shall now make a further reduction, eliminating the scheme $Fx = Ax + Bx$ from the preceding theorem, a suitable change in the initial functions being made. It will be convenient to introduce the following notation. If $Fx = Ax + Bx$ and $Gx = C(Ax, Bx)$, we shall denote F and G by $A + B$ and $C(A, B)$.

THEOREM 4. *All general recursive functions of one variable can be obtained by starting with a certain two primitive recursive functions and repeatedly using the formulas*

$$Fx = BAx, \qquad Fx = B^{-1}x$$

to construct a new function from known functions A and B. The latter formula is used only when B assumes all values.

PROOF. We first make the observation that any finite set of initial functions can be replaced by two initial functions. Indeed, the functions $F_1, F_2, \cdots, F_n$ are evidently obtainable by substitution from the two functions, K and

$$J(L, J(F_1, J(F_2, \cdots, J(F_{n-1}, F_n) \cdots))).$$

Thus we need only show that a finite set of primitive recursive initial functions is possible.

To each general recursive function F, we make correspond another function F^* by the formula

$$F^* = J(FK, L),$$

where J, K, and L are the Cantor pairing functions. We know that F can be generated from SO and K by addition, substitution, and inversion. Each step of this construction can be mirrored for the star functions in such a way that addition is unnecessary. Since

$$KF^*J(I, I) = KJ(FK, L)J(I, I) = KJ(F, I) = F,$$

we see that the function F itself can then be recovered.

We notice first that

$$B^*A^* = J(BK, L)J(AK, L) = J(BAK, L) = (BA)^*.$$

If B assumes all values, then B^* does also, and we have

$$\begin{aligned}(B^*)^{-1}x &= \mu z\{B^*z = x\} = \mu z\{J(BKz, Lz) = x\} \\ &= \mu z\{BKz = Kx \wedge Lz = Lx\} = J(\mu y\{By = Kx\}, Lx) \\ &= J(B^{-1}Kx, Lx) = (B^{-1})^*x.\end{aligned}$$

We have used the fact that J is a one-to-one monotone pairing function. Finally, we shall show that $(A+B)^*$ is obtained from A^* and B^* by substitution, making use of the special functions

$$G_1 = J(K, I), \qquad G_2 = J(KL, I), \qquad G_3 = J(K + KL, LLL).$$

Indeed we have

$$\begin{aligned}A^*G_1 &= J(AK, L)J(K, I) = J(AK, I), \\ B^*G_2 &= J(BK, L)J(KL, I) = J(BKL, I), \\ G_3B^*G_2 &= J(K + KL, LLL)J(BKL, I) = J(BKL + K, LL), \\ G_3B^*G_2A^*G_1 &= J(BKL + K, LL)J(AK, I) = J(BK + AK, L),\end{aligned}$$

that is,

$$(A + B)^* = G_3B^*G_2A^*G_1.$$

Thus if F is any general recursive function, the corresponding function F^* can be obtained from the two initial functions $(SO)^* = J(SO, L)$ and $K^* = J(KK, L)$ by use of substitution and inversion only, making use of the three auxiliary functions G_1, G_2, and G_3.

Finally, F itself is obtained by substitution from F^*, K, and $J(I, I)$. Thus all general recursive functions of one variable may be obtained by substitution and inversion from the seven functions,

$$K,\ J(I, I),\ J(SO, L),\ J(KK, L),\ J(K, I),\ J(KL, I),\ J(K + KL, LLL).$$

As already remarked, these seven functions can be replaced by two suitable functions, so the proof of the theorem is complete.

We shall however investigate the possibility of replacing these seven functions by fewer functions of a simpler nature than one of the two obtained by the above process. We start by proving a lemma valid for arbitrary pairing functions.

LEMMA. *If M and N can be obtained by substitution from I, K, and L, then $J(M, N)$ can be obtained by substitution from $J(LK, KL)$ and $J(L, I)$.*

PROOF. We notice first that

$$J(AK, BL)J(Cx, Dx) = J(AKJ(Cx, Dx), BLJ(Cx, Dx)) = J(ACx, BDx).$$

Thus

$$J(AK, BL)J(C, D) = J(AC, BD).$$

This and similar reductions will be used freely below. In particular, we observe the identities

$$J(L, K)J(A, B) = J(B, A),$$
$$J(KK, L)J(A, B) = J(KA, B),$$
$$J(LK, L)J(A, B) = J(LA, B).$$

From these it is clear that all required functions $J(M, N)$ can be obtained by substitution from the four functions

$$J(I, I), \qquad J(L, K), \qquad J(KK, L), \qquad J(LK, L).$$

It remains to show that these four functions can in turn be obtained from the two functions

$$G = J(LK, KL), \qquad H = J(L, I).$$

The following computation shows that this is possible.

$$GH = J(LK, KL)J(L, I) = J(LL, K),$$
$$GH^2 = J(LL, K)J(L, I) = J(L, L),$$
$$GH^3 = J(L, L)J(L, I) = J(I, I),$$
$$G^2H^3 = J(LK, KL)J(I, I) = J(L, K),$$

$$G^2H^4 = J(L, K)J(L, I) = J(I, L),$$
$$G^3H^4 = J(LK, KL)J(I, L) = J(L, KL),$$
$$G^3H^5 = J(L, KL)J(L, I) = J(I, K),$$
$$G^4H^5 = J(LK, KL)J(I, K) = J(L, KK),$$
$$G^2H^3G^4H^5 = J(L, K)J(L, KK) = J(KK, L),$$
$$GHG^2H^3 = J(LL, K)J(L, K) = J(LK, L).$$

THEOREM 5. *All general recursive functions of one variable can be obtained by substitution and inversion from the three functions* $J(LK, KL)$, $J(I, SO)$, $J(K+KL, I)$.

PROOF. It is sufficient to show that we can define the seven functions listed at the end of the proof of Theorem 4. In the first place,

$$J(K + KL, I)J(I, SO) = J(I + KSO, J(I, SO)) = J(I, J(I, SO)),$$
$$J(LK, KL)J(I, J(I, SO)) = J(L, I).$$

Hence by the lemma we can obtain all functions $J(M, N)$ where M and N are obtained from I, K, L by composition. Since the pairing function J is assumed one-to-one, we obtain $K = J(KK, LK)$. The remaining functions are obtained in virtue of the equations

$$J(K + KL, LLL) = J(K, LLLL)J(K + KL, I),$$
$$J(SO, L) = J(L, LK)J(I, SO).$$

We shall now show that it is impossible to reduce the two initial functions in Theorem 4 to a single function of any sort.

THEOREM 6. *There is no function A from which all general recursive functions can be obtained by substitution and inversion.*

PROOF. Such a function A would have to assume all values. For otherwise no function assuming all values could be obtained by substitution, and inversion would be impossible. If A were a one-to-one mapping of the integers onto themselves, then only such permutations would be definable. Hence A must assume some value more than once.

We shall show that the only functions assuming all values which are obtainable from A are $A^0 = I$, A, A^2, A^3, $\cdots$, and hence inversion is possible only for powers of A. Clearly, it is sufficient to show that

$$F = A^k(A^l)^{-1}A^m(A^n)^{-1} \cdots A^p(A^q)^{-1}$$

assumes all values only if it is a power of A. If $k < l$, then F does not assume all values, since $A^k(A^l)^{-1}$ does not. Indeed, we know that

A^{l-k} assumes some value at least twice; hence if $A^k(A^l)^{-1}$ assumed all values, then $I=A^{l-k}A^k(A^l)^{-1}$ would have to assume some value twice. Hence $k \geqq l$, and then $A^k(A^l)^{-1}=A^{k-l}A^l(A^l)^{-1}=A^{k-l}$. Thus F can be simplified. Repeating the argument, we find that if F assumes all values, it must be a power of A. Since no permutations are obtained except the identity, we do not obtain all general recursive functions.

5. **Representation of general recursive functions in terms of primitive recursive functions.** Starting from the Herbrand-Gödel metamathematical definition of general recursive function, Kleene showed that every such function can be expressed in the form

$$F\mathfrak{x} = A\mu y\{B(\mathfrak{x}, y) = 0\},$$

where A and B are primitive recursive functions and the required y exists for each x. Starting from the definition of this paper, we shall prove the possibility of such a representation, without using metamathematical arguments. We may also replace μ by ι in this formula.

THEOREM. *Every general recursive function F can be expressed in the form*

$$F\mathfrak{x} = A\iota y\{B(\mathfrak{x}, y) = 0\}$$

where A and B are primitive recursive functions, and for every $\mathfrak{x}$ there is a unique y such that $B(\mathfrak{x}, y)=0$.

PROOF. By successive applications of the pairing functions, we can reduce the theorem to the special case $\mathfrak{x}=x$. We can then apply the results of §4 concerning general recursive functions of one variable. We notice first that if F is primitive recursive, the representation

$$Fx = F\iota y\{|x - y| = 0\}$$

may be used. Thus according to Theorem 4 of §4 we need only check that the set of functions which can be represented in the form stated in the theorem is closed under substitution and inversion.

Suppose that

$$Fx = A\iota y\{B(x, y) = 0\}, \qquad Gx = C\iota y\{D(x, y) = 0\},$$

where A, B, C, D are primitive recursive. Then

$$\begin{aligned} FGx &= A\iota z\{B(Gx, z) = 0\} \\ &= A\iota z\{B(C\iota y\{D(x, y) = 0\}, z) = 0\}. \end{aligned}$$

In this formula, y and z are uniquely determined by x. Putting $w = J(y, z)$ (J is taken to be Cantor's pairing function), we see that

$$FGx = AL\iota w\{B(CKw, Lw) = 0 \wedge D(x, Kw) = 0\},$$

which reduces to the required form. Thus the set of functions considered is closed under substitution.

Now consider inversion. Again suppose that

$$Fx = A\iota y\{B(x, y) = 0\}.$$

Let us assume first $B(x, y) = 0$ determines y as a monotone non-decreasing function of x. The inverse function is determined by the formula

$$F^{-1}z = \mu x\{Fx = z\} = \mu x\{A\iota y\{B(x, y) = 0\} = z\}.$$

Now y is determined uniquely by x, and increases with x, hence the same is true of $w = J(x, y)$. Since the smallest possible w corresponds to the smallest possible x, we see that

$$F^{-1}z = K\mu w\{ALw = z \wedge B(Kw, Lw) = 0\}.$$

This can be reduced to the required form by the method of §2.

If B does not determine y as a monotone function of x, we can find a new representation of F with this property. Notice first that for a given x, we can choose u so that Ku, KLu, $\cdots$, KL^xu all have preassigned values, and the smallest possible u increases with each of them. Hence

$$Fx = AKL^x\mu u\{B(0, Ku) = 0 \wedge B(1, KLu) = 0 \wedge \cdots \wedge B(x, KL^xu) = 0\},$$

or

$$Fx = AKL^x\mu u\{G(x, u) = 0\}$$

where

$$G(x, u) = \sum_{n=0}^{x} B(n, KL^nu).$$

Clearly, the smallest u such that $G(x, u) = 0$ increases with x, since more conditions are imposed on it. Putting $v = J(2^xu, L^xu)$, and noticing that we always have $J(2z, 0) \geqq J(z, z)$, we see that v also increases with x. It follows that

$$Fx = AKL\mu v\{G(x, [Kv/2^x]) = 0 \wedge Lv = L^x[Kv/2^x]\}.$$

The functions occurring in the braces are all primitive recursive functions of v and x, and the smallest v satisfying the condition increases

with x. We may now reduce to the desired form as in §2.

COROLLARY. *Any general recursive function of one variable can be expressed in the form AB^{-1}, where A and B are primitive recursive and B assumes all values.*

PROOF. The reduction to this form is made as in §2.

COROLLARY. *A function $F\mathfrak{x}$ (of n variables) is general recursive if and only if it has the form $F\mathfrak{x}=AC\mathfrak{x}$, where Ay and $0^{|y-Cx|}$ are primitive recursive functions (of one and $n+1$ variables, respectively).*[5]

PROOF. If F is general recursive, it has the form

$$F\mathfrak{x} = A\iota y\{B(\mathfrak{x}, y) = 0\},$$

where A and B are primitive recursive. Putting

$$C\mathfrak{x} = \iota y\{B(\mathfrak{x}, y) = 0\},$$

we see that

$$0^{|y-Cx|} = 0^{B(x,y)}$$

is primitive recursive. Conversely, suppose $F\mathfrak{x}=AC\mathfrak{x}$, where Ay and $0^{|y-Cx|}$ are primitive recursive. Then

$$F\mathfrak{x} = A\iota y\{0^{|y-Cx|} = 1\}$$

is general recursive.

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[5] Stated by A. Tarski in a seminar at the University of California in the summer of 1943. Also see Th. Skolem, *Some remarks on recursive arithmetic*, Norske Videnskabers Selskabs, Forhandlinger vol. 17 (1944) pp. 103–106.

CONVEXICAL BLANKETS

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1. Introduction. Suppose given a metric space $\mathfrak{S}$, a set $A \subset \mathfrak{S}$, and a function which correlates with each point z in A such a family of nonvacuous subsets of $\mathfrak{S}$ that there are members of this family of diameter arbitrarily small, shrinking down upon, but not necessarily containing, the point z. Such a function is called a *blanket*[1] with domain A. It is known that if a blanket possesses certain covering properties, then almost everywhere differentiation is possible with respect to it.[2]

Herein we restrict ourselves to the study of a particular kind of blanket in the euclidean space of n dimensions $\mathfrak{R}_n$. A blanket F of the type under consideration is so set up that for z in its domain, $F(z)$ is defined with the aid of a certain family of convex sets which we call a *generator*. It will be shown in §3 that if ϕ is a measure in $\mathfrak{R}_n$ of a rather arbitrary nature, then we can always determine a *subblanket* of F which is *ϕ-regular* and which may be used for purposes of differentiation. This is true even when the convex sets in the generator are arbitrarily thin. Results of a stronger nature are known[3] if ϕ is taken to be Lebesgue measure in $\mathfrak{R}_n$. In §4, we devote our attention to the construction of an example which serves to indicate certain limitations upon the results which one might hope to obtain.

2. Preliminaries. In this section we state a number of definitions and conventions which are needed subsequently. It proves convenient to keep our terminology and notation generally in accord with M.

Let us agree that $z \in A = (z \text{ is a member of } A) = (z \text{ is in } A)$; $(z \notin A) = (z \text{ is not a member of } A) = (z \text{ is not in } A)$; $(A \subset B) = (A \text{ is a subset of } B) = (B \supset A) = (B \text{ contains } A)$; the integer 0 and the null set are identical.

If a_t is a set for each $t \in B$, then

$$\sum_{t \in B} a_t = E_z[(z \in a_t) \text{ for some } t \in B].$$

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[1] Unfamiliar technical terms are defined in §2.

[2] See A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235. Hereinafter this will be referred to as M.

[3] See M, pp. 215–216.

If $\mathfrak{F}$ is a family, then

$$\sigma\mathfrak{F} = \sum_{\beta\in\mathfrak{F}} \beta.$$

Thus $t\in\sigma\mathfrak{F}$ if and only if t is a member of some set in $\mathfrak{F}$.

If $a_t \geqq 0$ for each t in a countable set B, then

$$\sum_{t\in B} a_t$$

also denotes the appropriate numerical sum. However, it will be clear from the context whether set theoretic or numerical summation is intended in a particular instance.

Throughout §2 and §3 we let n denote a fixed positive integer, and we denote by $\mathfrak{R}_n$ euclidean space of n dimensions. We let ρ be that function for which

$$\rho(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

whenever $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ are points in $\mathfrak{R}_n$; ρ is the metric of $\mathfrak{R}_n$. Henceforth such terms as distance, interior, closed, Borel, and so on, will refer to the metric ρ.

If $\beta\subset\mathfrak{R}_n$, then K_β will denote the characteristic function of β.

If r is a real number, x and y are points in $\mathfrak{R}_n$, then rx, $x+y$, and $x-y$ will be understood to have their usual vectorial meanings.

2.1. DEFINITION.

$$C(z, r) = \underset{t}{E}\,[\rho(t, z) \leqq r].$$

Clearly $C(z, r)$ is the closed sphere of center z and radius r whenever $z\in\mathfrak{R}_n$ and $r\geqq 0$.

2.2. DEFINITION. For $A\subset\mathfrak{R}_n$ we define

$$\operatorname{diam} A = \sup_{(x\in A)(y\in A)} \rho(x, y) \qquad \text{if } A \neq 0;$$

$$\operatorname{diam} A = 0 \qquad \text{if } A = 0.$$

2.3. DEFINITION. *ϕ measures* $\mathfrak{R}_n$ if and only if ϕ is such a function on

$$\underset{\beta}{E}\,[\beta \subset \mathfrak{R}_n] \quad \text{to} \quad \underset{t}{E}\,[0 \leqq t \leqq \infty]$$

that

$$\phi(A) = \sum_{\beta\in\mathfrak{F}} \phi(\beta)$$

whenever $\mathfrak{F}$ is a countable family for which $A\subset\sigma\mathfrak{F}\subset\mathfrak{R}_n$.

2.4. DEFINITION. The family $\mathfrak{U}$ is defined by: $\phi \in \mathfrak{U}$ if and only if ϕ so measures $\mathfrak{R}_n$ that

$$\phi(A + B) = \phi(A) + \phi(B)$$

whenever

$$\inf_{(x \in A)(y \in B)} \rho(x, y) > 0.$$

2.5. DEFINITION.

$$\mathfrak{B} = \mathfrak{U} \cdot \underset{\phi}{E} [\phi(\beta) < \infty \text{ whenever } \beta \text{ is a bounded subset of } \mathfrak{R}_n].$$

2.6. DEFINITION. A is ϕ-measurable if and only if ϕ so measures $\mathfrak{R}_n$ that

$$\phi(T) = \phi(AT) + \phi(T - A)$$

for each subset T of $\mathfrak{R}_n$. It is well known that all Borel sets are ϕ-measurable whenever $\phi \in \mathfrak{B}$.

2.7. DEFINITION.

$$\operatorname{sng} z = \underset{t}{E} [t = z].$$

Thus sng z is the set whose sole member is z.

2.8. DEFINITION. the limit notations

$$\operatorname*{limalinf}_{\mathfrak{F} \ni \beta \to z} f(\beta), \qquad \operatorname*{limalsup}_{\mathfrak{F} \ni \beta \to z} f(\beta)$$

are defined, respectively, as

$$\lim_{t \to 0+} \left(\inf_{\beta \in H(t)} f(\beta) \right), \qquad \lim_{t \to 0+} \left(\sup_{\beta \in H(t)} f(\beta) \right),$$

where

$$H(t) = \mathfrak{F} \cdot \underset{\beta}{E} [\operatorname{diam} (\beta + \operatorname{sng} z) < t].$$

We write

$$\operatorname*{limal}_{\mathfrak{F} \ni \beta \to z} f(\beta) = \operatorname*{limalsup}_{\mathfrak{F} \ni \beta \to z} f(\beta)$$

if and only if

$$\operatorname*{limalsup}_{\mathfrak{F} \ni \beta \to z} f(\beta) = \operatorname*{limalinf}_{\mathfrak{F} \ni \beta \to z} f(\beta).$$

2.9. DEFINITION. We say F is a *blanket* if and only if F is such a function that for z in its domain

(i) $z \in \mathfrak{R}_n$ and $F(z)$ is a family of nonvacuous subsets of $\mathfrak{R}_n$;
(ii) $\beta \in F(z)$ implies $(\operatorname{diam} \beta) < \infty$;

(iii) $$\inf_{\beta \in F(z)} \{\operatorname{diam} (\beta + \operatorname{sng} z)\} = 0.$$

2.10. DEFINITION. If F is a blanket with domain A then the set

$$\sum_{z \in A} F(z)$$

is called the *spread* of F.

2.11. DEFINITION. F is a *natural* blanket if and only if F is a blanket and $z \in \beta$ whenever $\beta \in F(z)$.

2.12. DEFINITION. F is a *Borelian* (*close*) blanket if and only if F is such a blanket that β is a Borel (closed) set whenever β is in the spread of F.

2.13. DEFINITION. If Δ is a non-negative function whose domain is a family of sets H, then corresponding to each β in H we define $\Delta:\beta$ as the set S for which $z \in S$ if and only if there is such a set α in H that

$$z \in \alpha, \quad \alpha\beta \neq 0, \quad \text{and} \quad \Delta(\alpha) \leqq 2\Delta(\beta).$$

2.14. DEFINITION. If F is a natural blanket, $\phi \in \mathfrak{B}$, and there is such a non-negative function Δ whose domain is the spread of F that

$$\operatorname*{limalsup}_{F(z) \exists \beta \to z} \left[\Delta(\beta) + \frac{\phi(\Delta:\beta)}{\phi(\beta)} \right] < \infty$$

for ϕ-almost all z in the domain of F, then F is said to be a *ϕ-regular* blanket.

2.15. DEFINITION. The family $\mathfrak{F}$ *covers* A if and only if $A \subset \sigma\mathfrak{F}$.

2.16. DEFINITION. The family $\mathfrak{F}$ *covers ϕ-almost all of* A if and only if $\phi \in \mathfrak{B}$ and $\phi(A - \sigma\mathfrak{F}) = 0$.

2.17. DEFINITION. If $\phi \in \mathfrak{B}$, F is a Borelian blanket with domain A, and the spread of F contains a countable disjointed family which covers ϕ-almost all of A, then F is a *ϕ-heavy* blanket.

2.18. DEFINITION. G is a *subblanket* of F if and only if F and G are such blankets that $G(z) \subset F(z)$ whenever z is in the domain of G.

2.19. DEFINITION. If F is such a blanket that each of its subblankets is ϕ-heavy, then F is a *ϕ-strong* blanket.

The following is proved in M.

2.20. THEOREM. *Every ϕ-regular close blanket is ϕ-strong.*

2.21. DEFINITION. A family $\mathfrak{H}$ is a *nest* if and only if ($\beta_1 \in \mathfrak{H}$ and $\beta_2 \in \mathfrak{H}$) implies ($\beta_1 \subset \beta_2$ or $\beta_2 \subset \beta_1$).

2.22. DEFINITION. If $\mathfrak{H}$ is any family of subsets of $\mathfrak{R}_n$, and $z \in \mathfrak{R}_n$,

then the *z-translate* of $\mathfrak{H}$ is that family of sets obtained by translating each member of $\mathfrak{H}$ by the vector z; it is denoted by $\mathfrak{H}_z$.

2.23. DEFINITION. $\mathfrak{H}$ is said to be a *generator* if and only if $\mathfrak{H}$ is such a nonvacuous nest that

(i) $\beta \in \mathfrak{H}$ implies that β is a bounded, closed, symmetrical convex set with interior points, whose centroid is at the origin;

(ii) $$\inf_{\beta \in \mathfrak{H}} (\operatorname{diam} \beta) = 0.$$

2.24. DEFINITION. For any set $\beta \subset \mathfrak{R}_n$ with centroid a, and any $r \geqq 0$, we agree that β^r is the set of points of the form $a + r(x-a)$, where $x \in \beta$.

2.25. DEFINITION. For any set $\beta \subset \mathfrak{R}_n$ we define β^∇ as the set of points of the form $2x-y$, where $x \in \beta$ and $y \in \beta$.

In case β is a convex set it is easy to see that $x \in \beta^\nabla$ is equivalent to: there exists a set β' which is a translate of β, for which $\beta \cdot \beta' \neq 0$ and $x \in \beta'$.

2.26. DEFINITION. $\mathfrak{F}$ is a *hive* whenever $\mathfrak{F}$ is such a family of bounded, closed, convex subsets of $\mathfrak{R}_n$ that

(i) $\beta \in \mathfrak{F}$ implies β has interior points;

(ii) ($\beta_1 \in \mathfrak{F}$ and $\beta_2 \in \mathfrak{F}$) implies (β_1 either contains or is contained in some translate of β_2).

2.27. DEFINITION. F is a *convexical* blanket if and only if F is a blanket and $\mathfrak{H}$ is such a generator that if z is in the domain of F then there is a number r such that

(i) $0 < r \leqq 1$;

(ii) $\alpha \in \mathfrak{H}_z$ implies $\alpha^r \subset \beta \subset \alpha$ for some $\beta \in F(z)$.

3. Some properties of convexical blankets. For the entirety of this section we shall let F denote an arbitrary convexical blanket whose domain is A, and whose 2.27 associated generator is $\mathfrak{H}$. L and ϕ will denote, respectively, n-dimensional Lebesgue measure, and an arbitrary member of the 2.5 family $\mathfrak{B}$.

It proves convenient to introduce the function $\bar{\phi}$, associated with ϕ, so defined that for $\beta \in \mathfrak{R}_n$, $\bar{\phi}(\beta)$ is the inf of numbers of the form $\phi(\alpha)$, where α is such an open set that $\beta \subset \alpha$. It is easy to check[4] that $\bar{\phi} \in \mathfrak{B}$ and

(I) $$\bar{\phi}(\beta) = \phi(\beta)$$

whenever β is a Borel set.

[4] See A. P. Morse and J. F. Randolph, *The ϕ-rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) p. 246, Theorem 3.12.

An obvious fact which we shall have occasion to use is embodied in the following statement.

3.1. LEMMA. *If β is a symmetrical convex subset of $\mathfrak{N}_n$ then $\beta^\nabla = \beta^s$.*

3.2. LEMMA. *If β_0 is a symmetrical closed convex subset of $\mathfrak{N}_n$ having interior points, $1 \leqq s$, and $\mathfrak{K}_0$ is such a disjointed family of sets that $\beta \in \mathfrak{K}_0$ implies that β is a translate of β_0 for which $\beta^s \cdot \beta_0^s \neq 0$, then $\mathfrak{K}_0$ has at most $(3s)^n$ members.*

PROOF. We suppose that $\mathfrak{K}_0$ has N members. We select an arbitrary set β belonging to $\mathfrak{K}_0$, recall our hypotheses, and use 2.25 and 3.1 to conclude that

$$\beta \subset \beta^s \subset (\beta_0^s)^\nabla = (\beta_0^s)^s = \beta_0^{3s};$$

hence from the arbitrary nature of $\beta \in \mathfrak{K}_0$ it follows that

$$\sigma\mathfrak{K}_0 \subset (\beta_0^s)^\nabla = \beta_0^{3s};$$

$$L(\sigma\mathfrak{K}_0) \leqq L(\beta_0^{3s}) = (3s)^n L(\beta_0).$$

Thus, keeping in mind that $\mathfrak{K}_0$ is disjointed, β_0 is closed, and

$$L(\beta) = L(\beta_0)$$

whenever $\beta \in \mathfrak{K}_0$, it is evident that

$$NL(\beta_0) = \sum_{\beta \in \mathfrak{K}_0} L(\beta) = L(\sigma\mathfrak{K}_0) \leqq (3s)^n L(\beta_0).$$

Since $L(\beta_0) \neq 0$, the desired result follows at once, and the lemma is proved.

Recalling 2.22, we define for $0 < r \leqq 1$, $m > 0$, and $k > 0$,

$$E_{r,m,k} = A \cdot \underset{s}{E} [\phi(\alpha^s) \geqq m\phi(\alpha^r) \text{ whenever } \alpha \in \mathfrak{H}_s \text{ and } (\operatorname{diam} \alpha) < k].$$

3.3. THEOREM. *If $0 < r \leqq 1$, $m > (81/r)^n$, and $k > 0$, then $\phi(E_{r,m,k}) = 0$.*

PROOF. We shall prove the theorem by contradiction; we therefore assume that $\phi(E_{r,m,k}) > 0$. We further suppose, with no loss of generality, that $E_{r,m,k}$ is bounded.

We let ϵ denote an arbitrary positive number, recall (I) above, and determine such a bounded open set D that

$$(1) \qquad E_{r,m,k} \subset D; \qquad \bar{\phi}(D) = \phi(D) \leqq \bar{\phi}(E_{r,m,k}) + \epsilon < \infty.$$

It is clear that one can find such a sequence of closed sets $C_1 \subset C_2 \subset C_3 \subset \cdots$ that $D = \sum_{i=1}^{\infty} C_i$. Since $\bar{\phi} \in \mathfrak{V}$, it follows that $\bar{\phi}(E_{r,m,k}) = \bar{\phi}(D \cdot E_{r,m,k}) = \lim_{i \to \infty} \bar{\phi}(C_i \cdot E_{r,m,k})$. Hence we may and do determine

such a closed set $C \subset D$ that

$$\bar{\phi}(A_0) \geqq \bar{\phi}(E_{r,m,k}) - \epsilon, \tag{2}$$

where $A_0 = C \cdot E_{r,m,k}$.

We denote by δ the obviously positive infimum of numbers of the form $\rho(x, y)$, where $x \in C$ and $y \in (\mathfrak{R}_n - D)$, let c denote the smaller of the numbers k and δ, and select a set $\beta_0 \in \mathfrak{H}$ for which $9 \text{ (diam } \beta_0) < c$.

We next introduce the families $\mathfrak{F}'$, $\mathfrak{G}'$, $\mathfrak{K}'$, and $\mathfrak{J}'$ defined, respectively, by:

(i) $\alpha \in \mathfrak{F}'$ if and only if α is a set obtained by translating β_0 by the vector z, where z is some point in A_0;

(ii) $\alpha \in \mathfrak{G}'$ if and only if $\alpha = \beta^r$, where $\beta \in \mathfrak{F}'$;

(iii) $\alpha \in \mathfrak{K}'$ if and only if $\alpha = \beta^{r/3}$, where $\beta \in \mathfrak{F}'$;

(iv) $\alpha \in \mathfrak{J}'$ if and only if $\alpha = \beta^9$, where $\beta \in \mathfrak{F}'$.

There is obviously such a one-to-one correspondence between the families $\mathfrak{F}'$, $\mathfrak{G}'$, $\mathfrak{K}'$, and $\mathfrak{J}'$ that to each $\beta \in \mathfrak{F}'$ there correspond the sets β^r, $\beta^{r/3}$, and β^9 belonging, respectively, to $\mathfrak{G}'$, $\mathfrak{K}'$, and $\mathfrak{J}'$.

We easily determine by induction such a family $\mathfrak{K} \subset \mathfrak{K}'$ that

(v) $\mathfrak{K}$ is disjointed;

(vi) $\alpha \in \mathfrak{K}'$ implies that α intersects some member of $\mathfrak{K}$.

We let $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{J}$ denote, respectively, those subfamilies of $\mathfrak{F}'$, $\mathfrak{G}'$, and $\mathfrak{J}'$ which correspond to $\mathfrak{K}$ in the one-to-one way just mentioned.

Since $\mathfrak{K}$ is disjointed, $\sigma\mathfrak{K}$ is bounded, and

$$L(\alpha) = (r/3)^n L(\beta_0) > 0$$

for each $\alpha \in \mathfrak{K}$, it is clear that $\mathfrak{K}$, $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{J}$ are finite families.

The proof is completed in three steps.

Step I. $A_0 \subset \sigma\mathfrak{G} \subset \sigma\mathfrak{J} \subset D$.

PROOF. That $\sigma\mathfrak{G} \subset \sigma\mathfrak{J} \subset D$ follows from the definitions (i), (ii), and (iv), together with the fact that $9 \text{ (diam } \beta_0) < c$. To show that $A_0 \subset \sigma\mathfrak{G}$, we assume the contrary and suppose that $z \in (A_0 - \sigma\mathfrak{G})$. In accordance with (i) and (iii) we find such a set $\beta \in \mathfrak{F}'$ that $\beta^{r/3} \in \mathfrak{K}'$ and $z \in \beta^{r/3}$. Using (vi) and 3.1 we determine such a set $\alpha \in \mathfrak{F}$ that

$$\alpha^{r/3} \in \mathfrak{K}; \qquad (\beta^{r/3})(\alpha^{r/3}) \neq 0; \qquad \beta^{r/3} \subset (\alpha^{r/3})^\nabla = \alpha^r.$$

Since $\alpha^r \in \mathfrak{G}$, we learn that $z \in \sigma\mathfrak{G}$, contradicting the assumption that $z \notin \sigma\mathfrak{G}$. This completes the proof of Step I.

Step II. If $0 < r \leqq 1$, $\beta \in \mathfrak{F}$, and $z \in \beta^9$, then z is a member of not more than $(81/r)^n$ members of $\mathfrak{J}$.

PROOF. We define $\mathfrak{J}_0$ as that subfamily of $\mathfrak{J}$ whose members intersect β^9, and let $\mathfrak{K}_0$ be that subfamily of $\mathfrak{K}$ which corresponds to $\mathfrak{J}_0$ in the one-to-one way previously mentioned. Putting $\beta_0' = \beta^{r/3}$,

$s=27/r$, and recalling that $\mathfrak{K}_0$ is disjointed, we see that Lemma 3.2 may be applied to conclude that $\mathfrak{K}_0$ and $\mathfrak{J}_0$ each have at most $(3s)^n = (81/r)^n$ members, so that z is in at most $(81/r)^n$ members of $\mathfrak{J}$, as was to be shown.

Step III. $\phi(E_{r,m,k})=0$.

PROOF. We observe with the help of Step II that

$$\sum_{\gamma\in\mathfrak{J}} K_\gamma(z) \leqq (81/r)^n$$

for each $z\in\mathfrak{K}_n$, recall that $\sigma\mathfrak{J}\subset D$, and conclude that

$$\begin{aligned}(3)\qquad \sum_{\gamma\in\mathfrak{J}} \bar{\phi}(\gamma) &= \sum_{\gamma\in\mathfrak{J}} \int_D K_\gamma(z)\,d\bar{\phi}(z) \\ &= \int_D \left\{\sum_{\gamma\in\mathfrak{J}} K_\gamma(z)\right\} d\bar{\phi}(z) \leqq (81/r)^n \bar{\phi}(D).\end{aligned}$$

Hence, using (2), Step I, the facts that (diam $\beta)<k$ and $\beta\in\mathfrak{H}_s$ for some $z\in E_{r,m,k}$ whenever $\beta\in\mathfrak{F}$, the definition of $E_{r,m,k}$, (I), (3), and (1), it follows that

$$\begin{aligned}(4)\qquad 0 &< m\phi(E_{r,m,k}) \leqq m\bar{\phi}(E_{r,m,k}) \leqq m\bar{\phi}(A_0) + m\epsilon \\ &\leqq m\bar{\phi}(\sigma\mathfrak{G}) + m\epsilon \leqq m\sum_{\beta\in\mathfrak{G}} \bar{\phi}(\beta) + m\epsilon = m\left\{\sum_{\alpha\in\mathfrak{F}} \bar{\phi}(\alpha^r)\right\} + m\epsilon \\ &\leqq \sum_{\gamma\in\mathfrak{J}} \bar{\phi}(\gamma) + m\epsilon \leqq (81/r)^n\bar{\phi}(D) + m\epsilon \\ &\leqq (81/r)^n\bar{\phi}(E_{r,m,k}) + (81/r)^n\epsilon + m\epsilon < \infty.\end{aligned}$$

The arbitrary nature of ϵ in (4) permits us to infer that

$$0 < m\bar{\phi}(E_{r,m,k}) \leqq (81/r)^n\bar{\phi}(E_{r,m,k}) < \infty,$$

from which, in turn, we obtain $m\leqq(81/r)^n$. This contradiction of the original hypotheses proves Step III and completes the proof of 3.3.

3.4. THEOREM. *If* $0<r\leqq 1$, *then*

$$\operatorname*{limalinf}_{\mathfrak{G}\text{ ,}\exists\beta\to z} \frac{\phi(\beta^q)}{\phi(\beta^r)} < \infty$$

for ϕ*-almost all* $z\in A$.

PROOF. We let

$$A' = \underset{z}{E}\left[\operatorname*{limalinf}_{\mathfrak{G}\text{ ,}\exists\beta\to z} \frac{\phi(\beta^q)}{\phi(\beta^r)} < \infty\right],$$

and select any number $m>(81/r)^n$. If $z\in(A-A')$, it is evident that either

$$\operatorname*{limal}_{\mathfrak{H}_z \ni \beta \to z} \frac{\phi(\beta^q)}{\phi(\beta^r)} = \infty$$

or else there exists such a number $k>0$ that $\phi(\beta^r)=0$ whenever $\beta \in \mathfrak{H}_z$ and $(\operatorname{diam} \beta) < k$. It is therefore easy to see that

$$(A - A') \subset \sum_{i=1}^{\infty} E_{r,m,1/i},$$

and application of Theorem 3.3 yields $\phi(A-A')=0$. The theorem is thus proved.

We choose a positive integer q for which

$$2 < (1 + 1/2^m)^q,$$

let $\mathfrak{A}$ be the family defined by

$$\mathfrak{A} = \sum_{z \in A} \mathfrak{H}_z,$$

and so define the function Δ with domain $\mathfrak{A}$ that

$$\Delta(\alpha) = [L(\alpha)]^q$$

for each $\alpha \in \mathfrak{A}$. Referring to Definitions 2.22, 2.23, and 2.26, we see that $\mathfrak{A}$ is a hive. Recalling 2.13 and 3.1, it follows[5] that for each $\alpha \in \mathfrak{A}$,

(II) $$\Delta{:}\alpha \subset \alpha^{\nabla\nabla} = \alpha^q.$$

In accordance with Definition 2.27, we may and do associate with each $z \in A$ such a number r_z, $0 < r_z \leqq 1$, that for each $\alpha \in \mathfrak{H}_z$ there exists some $\beta \in F(z)$ for which

(III) $$\alpha^{r_z} \subset \beta \subset \alpha.$$

Using Theorem 3.4 and (II), we define such a blanket H with domain A that $H(z) \subset \mathfrak{H}_z$ whenever $z \in A$ and

(IV) $$\operatorname*{limalsup}_{H(z) \ni \alpha \to z} \frac{\phi(\Delta{:}\alpha)}{\phi(\alpha^{r_z})} \leqq \operatorname*{limalsup}_{H(z) \ni \alpha \to z} \frac{\phi(\alpha^q)}{\phi(\alpha^{r_z})} < \infty$$

for ϕ-almost all $z \in A$.

Keeping (III) in mind, we are able to construct G, such a 2.18 sublanket of F with domain A, that to each β in the 2.10 spread of G there correspond a point z_β and a set α_β for which

(V) $$z_\beta \in A; \quad \beta \in G(z_\beta); \quad \alpha_\beta \in H(z_\beta); \quad \alpha_\beta^{r_{z_\beta}} \subset \beta \subset \alpha_\beta.$$

We now so define the function Δ_0 whose domain is $\mathfrak{B}$, the spread of G, that

$$\Delta_0(\beta) = \Delta(\alpha_\beta)$$

whenever $\beta \in \mathfrak{B}$. Using (II), (V), and the obvious fact that

[5] For a proof see M, p. 215, Theorem 7.2.

$$\sum_{\beta\in\mathfrak{B}} \operatorname{sng} \alpha_\beta \subset \mathfrak{A},$$

it is easily checked that

(VI) $$\alpha_\beta^{r_\beta} \subset \beta \subset \Delta_0{:}\beta \subset \Delta{:}\alpha_\beta$$

whenever $\beta\in\mathfrak{B}$. Finally, since

$$\sum_{\beta\in G(z)} \operatorname{sng} \alpha_\beta \subset H(z)$$

whenever $z\in A$, we may use (IV) and (VI) to conclude that

$$\operatorname{limalsup}_{G(z)\ni\beta\to z} \frac{\phi(\Delta_0{:}\beta)}{\phi(\beta)} \leqq \operatorname{limalsup}_{G(z)\ni\beta\to z} \frac{\phi(\Delta{:}\alpha_\beta)}{\phi(\alpha_\beta^{r_\beta})} \leqq \operatorname{limalsup}_{H(z)\ni\alpha\to z} \frac{\phi(\Delta{:}\alpha)}{\phi(\alpha^{r_\alpha})} < \infty$$

for ϕ-almost all $z\in A$, which 2.14 leads to the following result.

3.5. THEOREM. *There is a ϕ-regular subblanket of F with domain A.*

A redefinition of convexical blankets might be effected by deleting the word "symmetrical" in (i) of 2.23, as it applies to the definition of the generator $\mathfrak{H}$, and in turn F itself. However, it happens that such a change would bestow no additional generality upon F or the results obtained in Theorem 3.5; for if F were defined in terms of such a nonsymmetrical generator $\mathfrak{H}$, then in accordance with well known properties[6] of convex sets, a suitable symmetrical generator $\mathfrak{H}'$ could be defined in terms of $\mathfrak{H}$ which would still satisfy the requirements of Definition 2.27 and which could therefore serve as generator for F in place of $\mathfrak{H}$.

Although Theorem 3.5 guarantees the existence of a ϕ-regular subblanket of F with domain A, nevertheless, in order that the subblanket possess suitable properties for purposes of differentiation, it is necessary that F satisfy an additional requirement not mentioned in Definition 2.27. Specifically, we need to require that F be a 2.12 Borelian or close blanket. If F is Borelian, then our 3.5 subblanket is suitable for differentiation by virtue of M, Theorems 11.2, 11.3, and 8.12. If F is a close blanket, we can say even more, since from 2.20 we can then conclude that our 3.5 subblanket is 2.19 ϕ-strong, and hence M, Theorem 8.10 holds for the subblanket.

4. **A counterexample.** If $\phi\in\mathfrak{V}$ and $\mathfrak{H}$ is a generator, then in view of Theorem 3.4 and (IV) above, one might ask: For each r, $0<r\leqq 1$, must there exist a generator $\mathfrak{H}'\subset\mathfrak{H}$ for which

[6] Use T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik, vol. 3, 1934, p. 53, formula (1).

$$\operatorname{limalsup}_{\delta_z \exists \beta \to z} \frac{\phi(\beta')}{\phi(\beta'')} < \infty$$

for ϕ-almost all $z \in \mathfrak{R}_n$? It turns out that the answer to this question is negative. We shall construct a certain example in the straight line space $\mathfrak{R}_1$, and outline a proof of this last statement. Henceforth, ρ will be the usual metric for $\mathfrak{R}_1$, and such terms as length, closed, open, and so on, will have reference to ρ.

We suppose that I is a closed interval in $\mathfrak{R}_1$ of length $l > 0$, and let n be a positive integer. We so choose $2n-1$ equally spaced points $P_1, P_2, \cdots, P_{2n-1}$ in the interior of I that they divide I into $2n$ equal parts. Each odd numbered point $P_1, P_3, \cdots, P_{2n-1}$ is made the center of a closed interval of length l/n^2. The n reduced intervals thus constructed are evidently disjointed and all are contained in I itself. We let 0 be that function for which $0(n, m, I)$ is the mth such interval counting from the right, whenever m is such a positive integer that $1 \leqq m \leqq n$.

We begin our construction by selecting a closed interval I_0 in $\mathfrak{R}_1$, of unit length. Recalling 2.7, we define by induction the families

$$\mathfrak{F}_0 = \operatorname{sng} I_0; \qquad \mathfrak{F}_n = \sum_{I \in \mathfrak{F}_{n-1}} \left\{ \sum_{m=1}^{n} \operatorname{sng} 0(n, m, I) \right\},$$

for $n = 1, 2, 3, \cdots$; we define $\mathfrak{F} = \sum_{n=0}^{\infty} \mathfrak{F}_n$. It is clear that $\sigma\mathfrak{F}_1 \supset \sigma\mathfrak{F}_2 \supset \sigma\mathfrak{F}_3 \supset \cdots$. We define the closed set $A_0 = \prod_{n=0}^{\infty} (\sigma\mathfrak{F}_n)$. It is to be noted that if n is a positive integer and $I \in \mathfrak{F}_n$, then there exists precisely one integer m and one interval $I' \in \mathfrak{F}_{n-1}$, such that $1 \leqq m \leqq n$ and $I = 0(n, m, I')$. We define inductively the function χ with domain $\mathfrak{F}$ for which

$$\chi(I_0) = 1; \qquad \chi\{0(n, m, I)\} = \frac{\lambda_n}{m^m} \chi(I)$$

whenever $I \in \mathfrak{F}_{n-1}$, m is such a positive integer that $1 \leqq m \leqq n$, and

$$\lambda_n = 1 \Big/ \sum_{j=1}^{n} \frac{1}{j^j}.$$

From this definition we see that

$$\sum_{m=1}^{n+1} \chi\{0(n+1, m, I)\} = \chi(I)$$

whenever $I \in \mathfrak{F}_n$. Applying induction to this last equation, it is easy to see that if $\mathfrak{G}_n \subset \mathfrak{F}_n$, n' is any integer exceeding n, and

$$\mathfrak{G}_{n'} = \mathfrak{F}_{n'} \cdot E_I[I \subset \sigma\mathfrak{G}_n],$$

then

(i) $$\sum_{t\in\mathfrak{G}_n}\chi(t)=\sum_{t'\in\mathfrak{G}_{n'}}\chi(t').$$

We define $\mathfrak{O}$ as the family of open subsets of $\mathfrak{R}_1$. For each positive integer n, we define ω_n as the function on $\mathfrak{O}$ for which

$$\omega_n(T)=\sum_{I\in\mathfrak{H}_n}\chi(I),$$

whenever $T\in\mathfrak{O}$ and

$$\mathfrak{H}_n=\mathfrak{F}_n\cdot\underset{I}{E}\,[I\subset T].$$

It is clear from (i) that $\omega_{n+1}(T)\geqq\omega_n(T)$ for each such n and T. We are therefore able to define that function Ω with domain $\mathfrak{O}$ for which

$$\Omega(T)=\lim_{n\to\infty}\omega_n(T)$$

whenever $T\in\mathfrak{O}$. Finally, we define Ω_0 as that function whose domain is the family of subsets of $\mathfrak{R}_1$, such that for $\beta\subset\mathfrak{R}_1$,

$$\Omega_0(\beta)=\inf_{T\in\mathfrak{G}}\Omega(T),$$

where

$$\mathfrak{G}=\mathfrak{O}\cdot\underset{T}{E}\,[\beta\subset T].$$

The following unproved statements are consequences of the foregoing definitions and observations.

(ii) If $I\in\mathfrak{F}$, then $\Omega_0(I)=\chi(I)$.

(iii) Ω_0 is a member of the 2.5 family $\mathfrak{B}$.

(iv) $\Omega_0(A_0)=1$.

For $n=0, 1, 2, 3, \cdots$ we let l_n denote the length of an interval belonging to $\mathfrak{F}_n$. We also define β_n as that interval with center at the origin of length

$$l_n\left(\frac{1}{n+1}+\frac{2}{(n+1)^2}\right).$$

We let $\mathfrak{H}$ be the 2.23 generator whose members are $\beta_0, \beta_1, \beta_2, \cdots$. It can be shown that if $0<r<1$ and $\mathfrak{H}'$ is any generator contained in $\mathfrak{H}$, then

$$\underset{\mathfrak{H}'\ni\beta\to z}{\operatorname{limalsup}}\frac{\Omega_0(\beta)}{\Omega_0(\beta')}=\infty$$

for Ω_0-almost all $z\in A_0$. This result is more than sufficient to justify the assertion made at the beginning of this section.

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ON CERTAIN SPECIAL SETS OF ORTHOGONAL POLYNOMIALS

G. SZEGÖ

In three recent notes in C. R. Acad. Sci. Paris (vol. 228 (1949) pp. 1363–1365, 1553–1556, 1998–2000) Mr. F. Pollaczek has introduced certain sets of orthogonal polynomials with remarkable properties. In the present paper we are considering primarily the polynomials introduced in the first note, the others being certain generalizations. In view of their relatively simple definition, these polynomials can readily be studied. They serve as illustrations for certain "irregular" phenomena in the theory of orthogonal polynomials.

Extensions of the well known integral representations (integrals of Laplace and Mehler) and of other formal properties will be discussed at a later opportunity.

1. Definitions. Theorems

1. The polynomials $P_n(x; a, b)$ introduced in the first note of Mr. F. Pollaczek depend on two real parameters a, b, $a \geqq |b|$. They can be defined by their generating series

$$\begin{aligned} f(x, z) &= f(\cos\theta, z) = \sum_{n=0}^{\infty} P_n(x; a, b) z^n \\ &= (1 - ze^{i\theta})^{-1/2+i\phi(\theta)}(1 - ze^{-i\theta})^{-1/2-i\phi(\theta)} \end{aligned} \tag{1}$$

where

$$\phi(\theta) = \frac{a\cos\theta + b}{2\sin\theta}. \tag{2}$$

The alternate form

$$\begin{aligned} f(x, z) &= (1 - 2xz + z^2)^{-1/2} \exp\left\{i\phi(\theta) \log\frac{1 - ze^{i\theta}}{1 - ze^{-i\theta}}\right\} \\ &= (1 - 2xz + z^2)^{-1/2} \exp\left\{(ax + b)\sum_{m=1}^{\infty} \frac{z^m}{m} U_{m-1}(x)\right\}, \end{aligned} \tag{3}$$

$$U_{m-1}(\cos\theta) = \frac{\sin m\theta}{\sin\theta},$$

shows that $P_n(x; a, b)$ is indeed a polynomial of degree n. For $a = b = 0$ we obtain the Legendre polynomials.

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The basic property of these polynomials is the following orthogonality relation

$$\int_{-1}^{+1} P_n(x; a, b)P_m(x; a, b)w(x; a, b)dx = \frac{\delta_{nm}}{n + (a + 1)/2} \tag{4}$$

where the weight-function is defined by

$$w(\cos\theta; a, b) = \frac{e^{(2\theta-\pi)\phi(\theta)}}{\cosh(\pi\phi(\theta))}. \tag{5}$$

In §2 we give a proof of the relation (4) which is simpler than that of Mr. F. Pollaczek. It is based (instead of complex integration) on the use of Laplace transforms.

2. We have the identity

$$w(-x; a, b) = w(x; a, -b). \tag{6}$$

Let $a > |b|$. Obviously, $w(x; a, b)$ becomes zero in exponential manner at the end points $x = -1$ and $x = 1$ of the interval of orthogonality. More precisely, $x = \cos\theta$,

$$w(x; a, b) \cong 2\exp\{(a + b)(1 - \pi/\theta\} \qquad \text{as } \theta \to 0 \tag{7}$$

so that log w *is not integrable in* $0 < \theta < \pi$.

3. The Jacobi polynomials associated with the weight

$$w(x) = (1 - x)^\alpha(1 + x)^\beta \tag{8}$$

offer a good indication how the behavior of the weight-function in the vicinity of the end points of the basic interval affects certain properties of the associated orthogonal polynomials. Let $\{p_n(x)\}$ be the *normalized* Jacobi polynomials, that is,

$$\int_{-1}^{+1} (p_n(x))^2 w(x)dx = 1. \tag{9}$$

Then[1]

(a) $p_n(1) \sim n^{\alpha+1/2}$,

(b) $p_n(x) \sim (x + (x^2 - 1)^{1/2})^n$,

(c) $p_n(\cos\theta) = A(\theta)\cos(n\theta + B(\theta)) + \epsilon_n(\theta)$, $\lim \epsilon_n(\theta) = 0$, $0 < \theta < \pi$.

---

[1] Cf. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, (4.1.1), p. 57, (8.21.9) and (8.21.10), p. 190, (4.3.3), p. 67. In the following we refer to this book as OP.

The sign $\sim$ indicates that the ratio of the given quantities approaches a limit different from 0. (We use the sign $\cong$ if this limit is 1.) The variable x is confined to the complex plane cut along the segment $-1, 1$ and $(x^2-1)^{1/2}>0$ for $x>1$. The functions $A(\theta)$, $B(\theta)$ are independent of n and analytic in $0<\theta<\pi$, $A(\theta)\neq 0$.

4. It is interesting to compare (a)–(c) with the corresponding properties of the orthogonal polynomials $P_n(x; a, b)$ defined above. Let $a>|b|$; we write

$$(n+(a+1)/2)^{1/2}P_n(x; a, b) = p_n(x; a, b) \tag{10}$$

so that $\{p_n(x; a, b)\}$ is the orthonormal set associated with the weight-function $w(x; a, b)$. Now,

$$P_n(1; a, b) = L_n(-a-b) \tag{11}$$

where L_n is the Laguerre polynomial so that (OP, Theorem 8.22.3, p. 193):

(a′) $p_n(1; a, b)\sim n^{1/4}\exp\{2(a+b)^{1/2}n^{1/2}\}$,

(b′) $p_n(x; a, b)\sim n^K(x+(x^2-1)^{1/2})^n$, $K=K(x)=(ax+b)/2(x^2-1)^{1/2}$,

(c′) $p_n(\cos\theta; a, b)=A_1(\theta)\cos(n\theta-\phi(\theta)\log n+B_1(\theta))+\epsilon_n(\theta)$

where $(x^2-1)^{1/2}>0$ for $x>1$ and $A_1(\theta)$, $B_1(\theta)$, $\epsilon_n(\theta)$ are functions of the same kind as in (c).

We observe the difference in the asymptotic behavior displayed by the formulas (a), (a′), (b), (b′), and (c), (c′). According to OP, Theorem 12.1.2 (p. 290) a formula of the type (b) holds whenever $w(x)$ is an arbitrary weight-function such that $\log w(\cos\theta)$ is integrable in $0, \pi$. According to 2. this is not the case for our weight-function $w(x; a, b)$.

As mentioned, the subject of §2 is a new proof of the orthogonality relation (4). In §3 we discuss the asymptotic relations (b′), (c′). §4 deals with a class $P_n^{(\lambda)}(x; a, b)$ of polynomials related to the ultraspherical polynomials $P_n^{(\lambda)}(x)$ (OP, 4.7, p. 80) in the same way as the polynomials $P_n(x; a, b)$ are related to the Legendre polynomials.

2. Proof of the orthogonality relation

Let z_1 and z_2 be real, $|z_1|<1$, $|z_2|<1$. We have

$$(1-z_1e^{i\theta})(1-z_2e^{i\theta}) = e^{i(\theta-\pi/2)}(1-z_1z_2)\sin\theta(1+iH),$$

$$H=\frac{(1+z_1z_2)\cos\theta-(z_1+z_2)}{(1-z_1z_2)\sin\theta}, \tag{1}$$

so that, $x=\cos\theta$,

$$
\begin{aligned}
&f(x,z_1)f(x,z_2)w(x;a,b)\sin\theta \\
&\quad = e^{i(\theta-\pi/2)\cdot 2i\phi(\theta)}((1-z_1z_2)\sin\theta)^{-1}(1+iH)^{-1/2+i\phi(\theta)} \\
&\qquad\qquad \cdot(1-iH)^{-1/2-i\phi(\theta)}w(x;a,b)\sin\theta \\
&\quad = (\cosh(\pi\phi(\theta)))^{-1}(1-z_1z_2)^{-1}(1+iH)^{-1/2+i\phi(\theta)}(1-iH)^{-1/2-i\phi(\theta)}.
\end{aligned} \tag{2}
$$

Using the familiar formula

$$
m^{-\alpha} = (\Gamma(\alpha))^{-1}\int_0^\infty e^{-ms}s^{\alpha-1}ds, \qquad \Re m > 0, \Re\alpha > 0, \tag{3}
$$

we obtain for the integral I of (2), $0 \leqq \theta \leqq \pi$,

$$
\begin{aligned}
I &= (1-z_1z_2)^{-1}\int_0^\pi \{\cosh(\pi\phi(\theta))\Gamma(1/2-i\phi(\theta))\Gamma(1/2+i\phi(\theta))\}^{-1} \\
&\qquad \cdot \int_0^\infty\int_0^\infty e^{-(1+iH)s_1-(1-iH)s_2}s_1^{-1/2-i\phi(\theta)}s_2^{-1/2+i\phi(\theta)}ds_1ds_2\cdot d\theta \\
&= \pi^{-1}(1-z_1z_2)^{-1}\int_0^\infty\int_0^\infty e^{-s_1-s_2}(s_1s_2)^{-1/2} \\
&\qquad \cdot \int_0^\pi e^{-iH(s_1-s_2)-i\phi(\theta)(\log s_1-\log s_2)}d\theta\cdot ds_1ds_2.
\end{aligned} \tag{4}
$$

Interchanging s_1 and s_2 or replacing θ by $-\theta$ has the same effect on the last integrand so that

$$
\begin{aligned}
I &= \pi^{-1}(1-z_1z_2)^{-1}\iint_{s_1\geqq s_2} e^{-s_1-s_2}(s_1s_2)^{-1/2} \\
&\qquad \cdot \int_{-\pi}^{\pi} e^{-iH(s_1-s_2)-i\phi(\theta)(\log s_1-\log s_2)}d\theta\cdot ds_1ds_2.
\end{aligned} \tag{5}
$$

Now[2]

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\exp\left(-i\frac{A\cos\theta+B}{\sin\theta}\right)d\theta \\
&\qquad = \frac{1}{i}\int \exp\left(\frac{A(\zeta^2+1)+2B\zeta}{\zeta^2-1}\right)\zeta^{-1}d\zeta = 2\pi e^{-A}
\end{aligned} \tag{6}
$$

where the complex integration has to be extended over $|\zeta| = 1$ with *indentations* around $\zeta = \pm 1$. The contribution of the indentations vanishes provided $A \pm B \geqq 0$ which is indeed the case since

[2] This already proves the orthogonality since A depends only on z_1z_2.

$$
\begin{aligned}
A &= \frac{1+z_1z_2}{1-z_1z_2}(s_1-s_2)+\frac{a}{2}(\log s_1-\log s_2),\\
B &= -\frac{z_1+z_2}{1-z_1z_2}(s_1-s_2)+\frac{b}{2}(\log s_1-\log s_2).
\end{aligned}
\tag{7}
$$

Hence, $s_1=s_2e^{\sigma}$,

$$
\begin{aligned}
I &= 2(1-z_1z_2)^{-1}\iint_{s_1\geq s_2}\\
&\quad\cdot\exp\left\{-s_1-s_2-\frac{1+z_1z_2}{1-z_1z_2}(s_1-s_2)\right\}s_1^{-(1+a)/2}s_2^{-(1-a)/2}ds_1ds_2\\
&= 2(1-z_1z_2)^{-1}\int_0^{\infty}\int_0^{\infty}\\
&\quad\cdot\exp\left\{-s_2(e^{\sigma}+1)-\frac{1+z_1z_2}{1-z_1z_2}s_2(e^{\sigma}-1)\right\}e^{-\sigma(1+a)/2}e^{\sigma}ds_2d\sigma\\
&= 2(1-z_1z_2)^{-1}\int_0^{\infty}\left\{e^{\sigma}+1+\frac{1+z_1z_2}{1-z_1z_2}(e^{\sigma}-1)\right\}^{-1}e^{-\sigma(1+a)/2}e^{\sigma}d\sigma\\
&= \int_0^{\infty}\frac{e^{-\sigma(1+a)/2}}{1-e^{-\sigma}z_1z_2}d\sigma=\sum_{n=0}^{\infty}\frac{z_1^nz_2^n}{n+(a+1)/2}
\end{aligned}
\tag{8}
$$

which proves §1 (4).

3. Asymptotic formulas

1. The generating function $f(x, z)$ in §1 (1) has two distinct singular points $e^{\pm i\theta}$ unless $-1/2+i\phi(\theta)$ is an integer (or infinite, that is, $\theta=k\pi$, k integer).

Let $\Im\theta>0$. The next singular point to the origin is $e^{i\theta}=x-(x^2-1)^{1/2}$. Applying Darboux's method we obtain

$$
P_n(x; a, b)\cong(1-e^{2i\theta})^{-1/2+i\phi(\theta)}C_{-1/2-i\phi(\theta),n}(-e^{-i\theta})^n. \tag{1}
$$

Since

$$
\begin{aligned}
C_{-1/2-i\phi(\theta),n} &= (-1)^n\frac{\Gamma(n+1/2+i\phi(\theta))}{\Gamma(n+1)\Gamma(1/2+i\phi(\theta))}\\
&\cong\frac{(-1)^nn^{-1/2+i\phi(\theta)}}{\Gamma(1/2+i\phi(\theta))},
\end{aligned}
\tag{2}
$$

we find for the right-hand expression in (1)

$$(3)\qquad n^{-1/2}\cdot\frac{(1-e^{2i\theta})^{-1/2+i\phi(\theta)}}{\Gamma(1/2+i\phi(\theta))}\cdot n^{i\phi(\theta)}e^{-in\theta}.$$

This establishes §1 (b′).

The result holds also if $-1/2+i\phi(\theta)$ is an integer equal to m. This integer must be necessarily non-negative so that $f(x, z)$ has a single pole of order $m+1$ at $e^{i\theta}$. Indeed, let $\zeta=e^{i\theta}$, $|\zeta|<1$; we have

$$(a+2m+1)\zeta^2+2b\zeta+(a-2m-1)=0.$$

(We have $a+2m+1\neq 0$, otherwise $\zeta=-a/b$, $|\zeta|\geqq 1$.) If we assume that ζ is not real, the other root of this equation would be $\bar{\zeta}$ so that

$$|\zeta\bar{\zeta}|=\left|\frac{a-2m-1}{a+2m+1}\right|<1,\ m\geqq 0.$$

In case $\theta=it$ or $it+\pi$, $t>0$, it is easy to see that $-1/2+i\phi(\theta)\geqq -1/2$.

2. Let $0<\theta<\pi$. Applying Darboux's method again we find

$$P_n(\cos\theta;\ a,\ b)$$
$$=2\Re(1-e^{2i\theta})^{-1/2+i\phi(\theta)}C_{-1/2-i\phi(\theta),n}(-e^{-i\theta})^n+O(n^{-3/2})$$
$$=2n^{-1/2}\Re\frac{\exp\{(-1/2+i\phi(\theta))(\log(2\sin\theta)+i(\theta-\pi/2))-in\theta+i\phi(\theta)\log n\}}{\Gamma(1/2+i\phi(\theta))}$$
$$+O(n^{-3/2}).$$

This establishes §1 (c′).

4. A GENERALIZATION

1. Let $\lambda>-1$, a and b real, $a\geqq|b|$. We introduce the weight-function

$$(1)\quad w^{(\lambda)}(x;\ a,\ b)=\pi^{-1}2^{2\lambda-1}e^{(2\theta-\pi)\phi(\theta)}\,|\,\Gamma(\lambda+i\phi(\theta))\,|^2(1-x^2)^{\lambda-1/2}$$

where $x=\cos\theta$ and $\phi(\theta)$ has the same meaning as in §1 (2). The generating function

$$(2)\qquad f^{(\lambda)}(x,z)=\sum_{n=0}^{\infty}P_n^{(\lambda)}(x;\ a,\ b)z^n$$
$$=(1-ze^{i\theta})^{-\lambda+i\phi(\theta)}(1-ze^{-i\theta})^{-\lambda-i\phi(\theta)}$$

defines certain polynomials $P_n^{(\lambda)}(x;\ a,\ b)$. We have $P_n^{(1/2)}(x;\ a,\ b)=P_n(x;\ a,\ b)$ and $P_n^{(\lambda)}(x;\ a,\ b)$ reduces to the ultraspherical polynomials for $a=b=0$.

We prove the orthogonality relation

$$\int_{-1}^{+1} P_n^{(\lambda)}(x; a, b) P_m^{(\lambda)}(x; a, b) w^{(\lambda)}(x; a, b) dx = \frac{\Gamma(n+2\lambda)}{n!} \frac{\delta_{nm}}{n+\lambda+a/2}. \tag{3}$$

2. The proof follows a similar line as in §2 so that we can be brief. We have

$$f^{(\lambda)}(x, z_1) f^{(\lambda)}(x, z_2) w^{(\lambda)}(x; a, b) \sin\theta = \pi^{-1} 2^{2\lambda-1} (1 - z_1 z_2)^{-2\lambda} \cdot |\Gamma(\lambda + i\phi(\theta))|^2 (1 + iH)^{-\lambda+i\phi(\theta)} (1 - iH)^{-\lambda-i\phi(\theta)} \tag{4}$$

so that we obtain for the integral $I^{(\lambda)}$ of this expression:

$$I^{(\lambda)} = \pi^{-1} 2^{2\lambda-1} (1 - z_1 z_2)^{-2\lambda} \cdot \int_0^{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(1+iH)s_1 - (1-iH)s_2} s_1^{\lambda-1-i\phi(\theta)} s_2^{\lambda-1+i\phi(\theta)} ds_1 ds_2 \cdot d\theta. \tag{5}$$

As in the case $\lambda = 1/2$ we are led to

$$I^{(\lambda)} = 2^{2\lambda}(1 - z_1 z_2)^{-2\lambda} \iint_{s_1 \geq s_2} \exp\left\{-s_1 - s_2 - \frac{1 + z_1 z_2}{1 - z_1 z_2}(s_1 - s_2)\right\} \cdot s_2^{\lambda-1-a/2} s_1^{(\lambda-1+a/2)} ds_1 ds_2$$

$$= 2^{2\lambda}(1 - z_1 z_2)^{-2\lambda} \int_0^{\infty} \int_0^{\infty} \exp\left\{-s_2(e^{\sigma} + 1) - \frac{1 + z_1 z_2}{1 - z_1 z_2} s_2 (e^{\sigma} - 1)\right\} \cdot s_2^{2\lambda-1} e^{\sigma(\lambda-1-a/2)} e^{\sigma} ds_2 d\sigma$$

$$= 2^{2\lambda}(1 - z_1 z_2)^{-2\lambda} \Gamma(2\lambda) \int_0^{\infty} \left\{e^{\sigma} + 1 + \frac{1 + z_1 z_2}{1 - z_1 z_2}(e^{\sigma} - 1)\right\}^{-2\lambda} \cdot e^{\sigma(\lambda-1-a/2)} e^{\sigma} d\sigma$$

$$= \Gamma(2\lambda) \int_0^{\infty} (1 - e^{-\sigma} z_1 z_2)^{-2\lambda} e^{-\sigma(\lambda+a/2)} d\sigma$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{n!} \frac{z_1^n z_2^n}{n+\lambda+a/2}.$$

This establishes (3).

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THE CLASSICAL EXISTENCE THEOREM OF NONLINEAR ANALYTIC DIFFERENTIAL EQUATIONS

EARL A. CODDINGTON

Let $f_1, \cdots, f_n$, where $f_j = f_j(s, w_1, \cdots, w_n)$, be n power series in $w_1, \cdots, w_n$,

$$f_j = \sum_{k=0}^{\infty} \cdots \sum_{m=0}^{\infty} a_j^{k \cdots m}(s) w_1^k \cdots w_n^m, \tag{1}$$

where the coefficients $a(s)$ are uniformly almost periodic functions on some half-plane, say,

$$\sigma > 0 \qquad (s = \sigma + it), \tag{2}$$

which possess Fourier expansions

$$a(s) \sim \sum_p \alpha_p e^{-\lambda_p s}, \tag{3}$$

where the exponents λ_p are all positive and bounded away from zero,

$$\lambda_p \geqq \lambda_0 > 0. \tag{4}$$

Further, let $A_j^{k \cdots m} < \infty$, where

$$A_j^{k \cdots m} = \text{l.u.b.}_{\sigma > 0} \left| a_j^{k \cdots m}(s) \right|, \tag{5}$$

and assume that for some, sufficiently small, positive r,

$$\sum_{k=0}^{\infty} \cdots \sum_{m=0}^{\infty} A_j^{k \cdots m} r^{k + \cdots + m} < \infty. \tag{6}$$

In particular, (6) implies that the functions $f_1, \cdots, f_n$ are all regular in the domain

$$\sigma > 0, \ |w_1| < r, \cdots, |w_n| < r. \tag{7}$$

In these notations, the following theorem will be proved:

THEOREM. *In the domain* (7), *let* $f_1, \cdots, f_n$ *be regular functions of the form* (1) *with uniformly almost periodic coefficient functions defined on* (2) *having Fourier expansions of the type* (3), *satisfying* (4). *Then, if* (6) *holds, there exists a half-plane* $\sigma > l \geqq 0$ *on which the system of differential equations*

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$$dw_j/ds = f_j(s, w_1, \cdots, w_n) \qquad (j = 1, \cdots, n) \tag{8}$$

possesses a uniformly almost periodic solution vector $w_1(s), \cdots, w_n(s)$ *satisfying the "initial conditions"* $w_j(s) \to 0$ *as* $\sigma \to \infty$, $j = 1, \cdots, n$. *In addition, this uniformly almost periodic solution is the only solution satisfying the latter "initial conditions."*

The latter initial conditions will be proved by showing that a uniformly almost periodic solution has exponents which are minorized by the positive constant λ_0 occurring in (4). It will also be clear from the proof that the exponents occurring in the solution are linear combinations, with non-negative integral coefficients, of the exponents appearing in the expansions (3) of the coefficients of (1).

This theorem is an extension of the classical existence theorem for analytic systems of differential equations,

$$dw_j/dz = F_j(z, w_1, \cdots, w_n) \qquad (j = 1, \cdots, n)$$

with the initial condition $w_1(0) = 0, \cdots, w_n(0) = 0$ (z corresponds to e^{-s}). Another extension of the classical existence theorem was considered in [2],[1] for the case where the coefficient functions $a(s)$ in (1) are all absolutely convergent Laplace-Stieltjes transforms with a positive lower limit of integration,

$$a(s) = \int_{\lambda_0}^{\infty} e^{-s\lambda} d\alpha(\lambda), \qquad \lambda_0 > 0,$$

where $\sigma > 0$. Neither of these extensions of the classical theorem contains the other. The latter extension had to be proved by using the integral analogue of the method of comparison of coefficients and convolution theory, whereas the present proof will depend on an application of the method of successive approximations. The latter method was used in [3] for that particular case of the theorem where the system (8) becomes linear, a case in which, of course, more information was obtained concerning the existence region of the solution than in the present nonlinear case.

For simplicity the theorem will be proved for the scalar case, $n = 1$. Since the proof will depend upon majorants in the method of successive approximations, it will be clear that this is no loss of generality.

For $n = 1$, (8) reduces to

$$w' = f(s, w) \qquad (' = d/ds), \tag{9}$$

where

[1] Numbers in brackets refer to the references at the end of the paper.

$$f(s, w) = \sum_{m=0}^{\infty} a_m(s) w^m, \tag{10}$$

and, by virtue of (3) and (4), the $a_m(s)$ are uniformly almost periodic functions on (2) with Fourier expansions

$$a_m(s) \sim \sum^{m} \alpha_p e^{-\lambda_p s}, \qquad \lambda_p \geqq \lambda_0 > 0. \tag{11}$$

The assumption (6) is that for some, sufficiently small, $r>0$,

$$\sum_{m=0}^{\infty} A_m r^m < \infty , \tag{12}$$

where

$$A_m = \text{l.u.b.}_{\sigma>0} \left| a_m(s) \right| . \tag{13}$$

The assertion is that (12) implies that there exists a half-plane $\sigma > l \geqq 0$ on which (9) has a unique uniformly almost periodic solution $w(s)$ for which $w(s) \to 0$, $\sigma \to \infty$.

As to the value of l, the following information will be obtained: *If the lower bound* λ_0 *occurring in* (11) *is assumed to be* 1, *and the sum of the series* (12) *for* $r<1$ *to be less than* 1, *then* l *can be chosen to be* 0. The proof of this will depend upon the use of Bohr's compactness theorem of normal families of regular, uniformly almost periodic functions. In fact, the argument is the same as that used in [5] to obtain the best value of the radius of regularity of the solution of an implicit analytic system. It follows from examples, even in the classical case of ordinary power series (cf. [4]), that these values of the constants can not be improved.

PROOF OF THE THEOREM. The exponents λ_p can be assumed to satisfy

$$\lambda_p \geqq 1 \tag{14}$$

in (11), for it is assumed that $\lambda_p \geqq \lambda_0$ holds for all p, and the assumptions and assertions of the theorem remain unchanged if s is replaced everywhere by s/λ_0. Also it is no essential restriction on (12) to assume that $r<1$.

The assumption (14) implies that the functions $a_m(s)e^s$ are uniformly almost periodic on $\sigma>0$ with non-negative exponents, and consequently tend to finite limits as $\sigma \to \infty$ (cf. [1, p. 152]). This implies, by (13), that $\left| a_m(s) \right| \leqq A_m e^{-\sigma}$, $\sigma>0$. In particular, if $A_m(\sigma_0)$ denotes the least upper bound of $\left| a_m(s) \right|$ on the half-plane

$\sigma > \sigma_0 (A_m(0) = A_m)$, it follows that $A_m(\sigma_0) \leqq A_m e^{-\sigma_0}$. Therefore the series

$$\sum_{m=0}^{\infty} A_m(\sigma_0) r^m$$

can be made to have a sum less than 1 if σ_0 is chosen sufficiently large, and $r < 1$. Since the theorem asserts the existence of a solution only on *some* half-plane $\sigma > l \geqq 0$, it is clear, in view of the foregoing remarks, that it can be assumed at the outset that

$$\sum_{m=0}^{\infty} A_m r^m < 1, \quad r < 1. \tag{15}$$

Define the successive approximations $w_0(s)$, $w_1(s)$, $\cdots$ as follows:

$$w_0(s) \equiv 0, \quad w'_{k+1}(s) = f(s, w_k(s)), \qquad k = 0, 1, 2, \cdots, \tag{16}$$

and

$$\lim_{\sigma \to \infty} w_{k+1}(s) = 0. \tag{17}$$

First, it should be noted that if $w_k(s)$ has been defined in the half-plane $\sigma > 0$ in such a way that it is uniformly almost periodic and satisfies

$$|w_k(s)| < 1, \quad \sigma > 0, \tag{18}$$

and is such that all of its exponents satisfy (14), then (16) and (17) determine in the half-plane $\sigma > 0$ a uniformly almost periodic function $w_{k+1}(s)$ for which (18) is true when k is replaced by $k+1$, and for which the exponents satisfy (14). The truth of this statement follows from a lemma used in [3, p. 861] concerning the indefinite integral of a bounded uniformly almost periodic function on $\sigma > 0$. In fact if (18) is satisfied, then by (15), $|f(s, w_k)| < 1$, and that lemma asserts that $w_{k+1}(s)$ is uniformly almost periodic on $\sigma > 0$, $|w_{k+1}(s)| < e^{-\sigma} < 1$, $\sigma > 0$, by virtue of (14), and the Fourier exponents of $w_{k+1}(s)$ are those of $f(s, w_k)$.

It will now be shown that the series

$$\sum_{k=0}^{\infty} \{w_{k+1}(s) - w_k(s)\} \tag{19}$$

is uniformly convergent on the half-plane $\sigma > 0$. In order to show this, note that (15) implies that the partial derivative $f_w(s, w)$ is bounded on every half-plane $\sigma \geqq \delta > 0$ and, from (10) and (11), $f_w(s, w) \to 0$ as $\sigma \to \infty$, uniformly in t, for all $|w| < 1$ (cf. [1, p. 151]). Therefore, for

every constant $c<1$, there exists an $l \geqq 0$ such that

$$|f_w(s, w)| \leqq c < 1, \qquad \sigma > l, \ |w| < 1. \tag{20}$$

From this, and since $f(s, w)$ is a power series in w for which (15) is true, it follows that $f(s, w)$ satisfies a Lipschitz condition, that is, the inequality

$$|f(s, w^*) - f(s, w^{**})| \leqq c|w^* - w^{**}| \qquad (c < 1) \tag{21}$$

holds whenever (s, w^*), (s, w^{**}) are a pair of points in the domain

$$\sigma > l, \quad |w| < 1. \tag{22}$$

By virtue of (16) and (18) this implies the inequality

$$|w'_{k+1}(s) - w'_k(s)| \leqq c|w_k(s) - w_{k-1}(s)|, \qquad \sigma > l. \tag{23}$$

If

$$B_k = \text{l.u.b.}_{\sigma > l} |w_k(s) - w_{k-1}(s)|, \tag{24}$$

then from (23), (14), and the lemma referred to after (18), it follows that

$$|w_{k+1}(s) - w_k(s)| \leqq cB_k, \qquad \sigma > l. \tag{25}$$

This implies that

$$B_{k+1} \leqq cB_k. \tag{26}$$

Since $|w_1(s) - w_0(s)| \leqq A_0 e^{-\sigma}$, $\sigma > 0$, $B_1 \leqq A_0 e^{-l} \leqq A_0$, and therefore an induction on (26) shows that

$$B_{k+1} \leqq A_0 c^k. \tag{27}$$

But $c<1$, $\sigma > l$, and therefore (27), by virtue of (24), proves that the series (19) is absolutely-uniformly convergent for $\sigma > l$. Its sum, $w(s)$, is uniformly almost periodic for $\sigma > l$, since every term of (19) is.

To say that the series (19) is uniformly convergent is to say that $w_k(s) \to w(s)$, $k \to \infty$, uniformly for $\sigma > l$. But the sequence of regular functions $w_1(s)$, $w_2(s)$, $\cdots$ is uniformly bounded for $\sigma > 0$ (cf. (18)). Therefore, the $w_k(s)$ tend uniformly to the regular function $w(s)$ on every compact subset of $\sigma > 0$. Furthermore, from (18), $|w(s)| \leqq 1$, $\sigma > 0$. By a standard theorem in almost periodic functions (cf. [1, pp. 142, 143]) it follows, since $w(s)$ is uniformly almost periodic on $\sigma > l \geqq 0$, regular and bounded on $\sigma > 0$, that $w(s)$ is uniformly almost periodic on the half-plane $\sigma > 0$.

Furthermore, (25), (23), (16), (17) show that the derived series

$$\sum_{k=0}^{\infty} \{w'_{k+1}(s) - w'_k(s)\}$$

is uniformly convergent for $\sigma > 0$. Therefore $w(s)$ is differentiable on $\sigma > 0$ and, by virtue of (16), satisfies (9).

From (10) and (11) it follows that if a solution $w = w(s)$ of (9) tends to 0, as $\sigma \to \infty$, then $f(s, w(s)) = O(e^{-\sigma})$, $\sigma \to \infty$ (cf. (14)). This implies, by virtue of (9), that $w' = O(e^{-\sigma})$, $\sigma \to \infty$, and consequently $w = O(e^{-\sigma})$, $\sigma \to \infty$. Therefore, the last assertion of the theorem, concerning the uniqueness of the solution, follows from standard arguments used for differential equations satisfying a Lipschitz condition.

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CENTER OF FLEXURE OF A TRIANGULAR BEAM

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1. Introduction. The resolution of Saint-Venant's general problem of a loaded beam into the simpler problems of torsion and pure flexure involves the determination of a load point P in the end section such that the local twist vanishes at a point Q in the section. The position of P in general depends upon the position of Q, the choice of which in turn is governed by physical considerations. For example, the local twist may be required to vanish at the load point itself, in which case the point is commonly called the center of shear. For most problems the preferred position of Q is at the centroid, since for this choice the average value of the local twist is zero over the entire section. Moreover, when Q is thus chosen, the point P, called the center of flexure of the section, is closely related both to the center of least strain[1] and to the center of twist[2] as these terms are usually defined.

The center of flexure has been determined for only a few sections which have uniaxial symmetry. For sections in the form of an isosceles triangle only the cases of the right triangle[3] and the incompressible equilateral triangle[4] have been treated theoretically, while an experimental determination of the center of flexure of a narrow section has been made by Duncan, Ellis, and Scruton in connection with their studies on airfoils.[5] In this paper the flexure solution of B. R. Seth[6] is used to derive a formula for the center of flexure of any isosceles-triangular section for any value of Poisson's ratio. This formula contains as an unknown factor only the torsional rigidity, for which an approximation has been obtained by Duncan, Ellis, and Scruton by means of the Rayleigh-Ritz method.[7] With this approximation the formula gives a result in good agreement with the experimental determination of these authors.

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[3] Ibid. pp. 200–205.

[4] S. Timoshenko, *Theory of elasticity*, 1934, p. 300.

[5] W. J. Duncan, D. L. Ellis, and C. Scruton, The Philosophical Magazine (7) vol. 16 (1933) pp. 201–235.

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[7] Loc. cit. pp. 231–235.

2. Derivation of the formula. We consider a uniform cantilever beam in the form of a right cylinder of isosceles-triangular cross-section and take the z-axis to be that generator which contains the vertices enclosed by the equal sides of the sections (Fig. 1). The

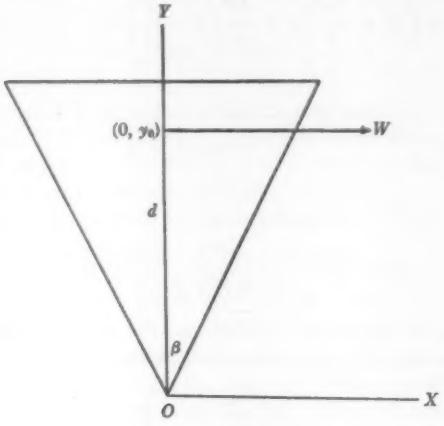

FIG. 1

origin of rectangular coordinates in the plane of the free end is at the apex of the triangular section, and the y-axis lies along the axis of symmetry. We denote by d the altitude and by β the semi-vertical angle of the triangle. A load of magnitude W is applied parallel to the x-axis from the point $(0, y_0)$. For this triangle the nonvanishing stress components which lie in the plane of the section are given by

$$(1) \qquad \tau_{xz} = \mu\tau\left(\frac{\partial\phi}{\partial x} - y\right) + \frac{\mu W}{EI}\left[\frac{\partial\chi}{\partial x} - \frac{\sigma x^2}{2} - \left(1 - \frac{\sigma}{2}\right)\left(y - \frac{2d}{3}\right)^2\right],$$

$$(2) \qquad \tau_{yz} = \mu\tau\left(\frac{\partial\phi}{\partial y} + x\right) + \frac{\mu W}{EI}\left[\frac{\partial\chi}{\partial y} - x(2+\sigma)\left(y - \frac{2d}{3}\right)\right].$$

Here I is the moment of inertia with respect to the y-axis, E is Young's modulus, σ is Poisson's ratio, μ is the modulus of rigidity, τ is a constant, ϕ is the torsion function, and χ the flexure function of the section. The lateral surface of the cylinder is supposed free from surface forces, this fact being expressed by the equation

$$(3) \qquad \tau_{xz}\cos(x, n) + \tau_{yz}\cos(y, n) = 0,$$

where n is the direction normal to the surface.

Seth has shown that the flexure function is given by

$$\chi = \left(A - \frac{4d}{3}\right)xy + \left[B - \frac{1}{3}\left(1 - \frac{\sigma}{2}\right)\right](x^3 - 3xy^2) + \left(H + \frac{2\sigma d}{3}\right)\phi + \frac{4d^2}{9}\left(1 - \frac{\sigma}{2}\right)x, \tag{4}$$

where

$$A = \frac{d[(1+\sigma)\tan^2\beta - \sigma](\tan^2\beta + 1)}{3\tan^2\beta - 1},$$

$$B = \frac{(1-2\sigma)\tan^2\beta}{3(3\tan^2\beta - 1)},$$

$$H = \frac{d[(1+\sigma)\tan^2\beta - \sigma](\tan^2\beta - 1)}{3\tan^2\beta - 1}.$$

According to Stevenson's definition, the center of flexure is that point of loading for which the local twist,

$$\frac{1}{2\mu}\left(\frac{\partial \tau_{ys}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right),$$

vanishes at the centroid of the section. Under this definition $(0, y_0)$ is the center of flexure provided $\tau = 0$. Equating the moment about the z-axis of the load W applied at $(0, y_0)$ to the torsional couple due to the stresses τ_{ys} and τ_{xz}, we have

$$-Wy_0 = \int_0^d \int_{-y\tan\beta}^{y\tan\beta} (x\tau_{ys} - y\tau_{xz})\,dx\,dy. \tag{5}$$

Substituting in (5) the values of τ_{ys} and τ_{xz} given by (1) and (2) with $\tau = 0$, using (4), and introducing the torsional rigidity D, defined by the equation

$$D = \mu \iint_S \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} + x^2 + y^2\right) dx\,dy, \tag{6}$$

we obtain after integration,

$$y_0 = \frac{2d\{5[(1+\sigma)\tan^2\beta - \sigma] - 2(1-2\sigma)\}}{5(1+\sigma)(3\tan^2\beta - 1)} - \frac{3(1+\sigma)\tan^4\beta - 3\tan^2\beta + \sigma}{(1+\sigma)\tan^2\beta(3\tan^2\beta - 1)}\,\frac{D}{\mu d^3}. \tag{7}$$

3. **Principal results.** It is clear from (7) that the torsional rigidity D must be known before y_0 can be determined. The exact values of D are known only for the isosceles right triangle and the equilateral triangle.[8] In the former case, using $D=0.10436\mu d^4$, we find for $\beta=\pi/4$,

$$\frac{y_0}{d} = .6000 - .00872\,\frac{\sigma}{1+\sigma},$$

which agrees with Stevenson's result for this section. For the equilateral triangle, $D=\mu d^4/15\cdot 3^{1/2}$. Substitution of this value in (7) with $\beta=\pi/6$ and $\sigma=1/2$ yields $y_0=2d/3$, showing that in this case the center of flexure coincides with the centroid.

For arbitrary σ we find for the equilateral triangle the limiting value

$$y_0 = \frac{11+8\sigma}{15(1+\sigma)}\,d. \tag{8}$$

This is a new result, which shows that despite the triaxial symmetry of the figure the center of flexure does not always coincide with the center of gravity. From (1), (2), and (4), using the torsion function for the equilateral triangle[9] adapted for the boundary shown in Fig. 1, the exact stresses are found to be

$$\tau_{xz}=\frac{\mu W}{6EI}\left[(1-2\sigma)(x^2-y^2)-6x^2+2dy\right], \quad \tau_{yz}=\frac{\mu W}{3EI}(1+4\sigma)(d-y)x$$

These satisfy the boundary condition (3) only on the side $y=d$. On the sides $y=\pm 3^{1/2}x$ the stresses satisfy the condition

$$\int_C \left[\tau_{xz}\cos(x,n)+\tau_{yz}\cos(y,n)\right]ds = 0,$$

where the integral is taken along the side. Unless the standard flexure problem is thus relaxed to the extent of permitting a resultant instead of an exact boundary condition to be satisfied on two of the sides, (8) has no significance. Under the generalized condition, however, the triangle still possesses a point which may be called the center of flexure. Its position is shifted from the centroid toward the side on which the exact condition is satisfied by an amount which depends on Poisson's ratio. If the triangle fails to be perfectly equilateral, as

[8] S. Timshenko, loc. cit. pp. 237, 251.

[9] I. S. Sokolnikoff, *Mathematical theory of elasticity*, 1946, p. 140.

is nearly always the case in practice, the exact conditions may be imposed, and the center of flexure will lie very near the centroid for all values of σ.

Equation (6) shows that D requires for its determination a knowledge of the torsion function ϕ. Although this function has been obtained by Seth for certain triangular sections,[10] it is convenient for values of β other than $\pi/4$ to use the approximate formula of Duncan, Ellis, and Scruton,

$$D = \frac{2\mu d^4 \tan^3\beta}{3(1-\tan^2\beta)} \cdot \frac{(6\tan^2\beta+10)^{1/2}-4\tan\beta}{(6\tan^2\beta+10)^{1/2}+4\tan\beta}.$$

With this value of D, y_0/d has been calculated for various values of β as follows:

β	y_0/d	β	y_0/d
$\frac{\pi}{24}$	$.7812 + .1163\frac{\sigma}{1+\sigma}$	$\frac{7\pi}{24}$	$.5726 - .0075\frac{\sigma}{1+\sigma}$
$\frac{\pi}{12}$	$.7401 + .0493\frac{\sigma}{1+\sigma}$	$\frac{\pi}{3}$	$.5495 - .0055\frac{\sigma}{1+\sigma}$
$\frac{\pi}{8}$	$.7060 + .0190\frac{\sigma}{1+\sigma}$	$\frac{3\pi}{8}$	$.5306 - .0038\frac{\sigma}{1+\sigma}$
$\frac{5\pi}{24}$	$.6436 - .0530\frac{\sigma}{1+\sigma}$	$\frac{5\pi}{12}$	$.5173 - .0018\frac{\sigma}{1+\sigma}$
$\frac{\pi}{4}$	$.6000 - .0083\frac{\sigma}{1+\sigma}$	$\frac{11\pi}{24}$	$.5090 - .0005\frac{\sigma}{1+\sigma}$

Fig. 2 displays in graphical form the dependence of the position of the center of flexure on σ for selected values of β. When the vertical angle is small, the position of the center of flexure is displaced from the centroid toward the base with increasing σ, while the displacement is in the opposite direction when β is greater than $\pi/6$. The shift is most pronounced for small angles. For angles greater than $\pi/4$ the dependence of the center of flexure on σ is negligible, varying from 0.5% for $\beta=\pi/4$ to 0.04% for $\beta=11\pi/24$. The practice of

[10] B. R. Seth, Quart. J. Math. Oxford Ser. vol. 5 (1934) pp. 170–171.

choosing a convenient value of σ arbitrarily in the solution of particular problems is thus theoretically justified at least for certain cases.

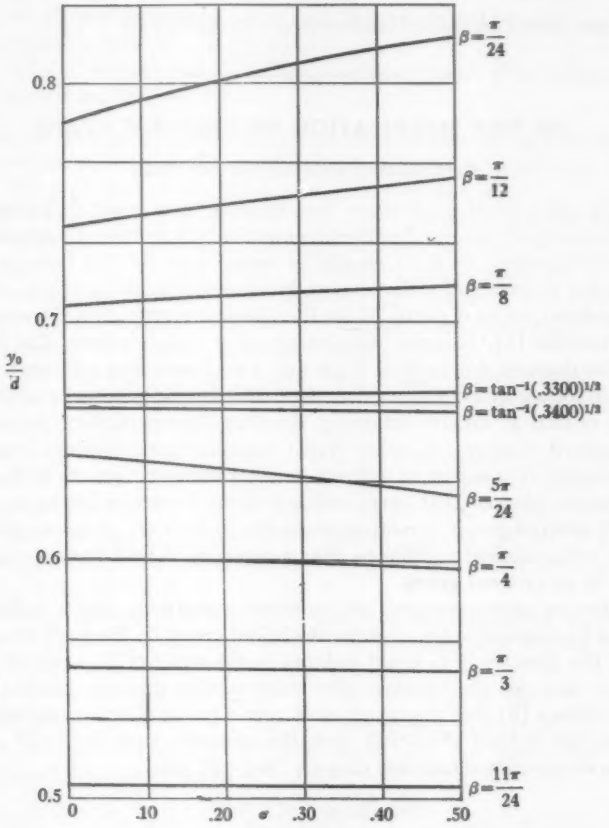

FIG. 2

As a final verification of the validity of (7), we may compare its prediction with the experimental results of Duncan, Ellis, and Scruton on the center of flexure for a steel prism whose cross-section is an isosceles triangle of semi-vertical angle nearly 0.041 radian in magnitude. The measurements, obtained for the single value $\sigma = 0.27$,

were made on sections at various distances from the fixed end of the beam. As extreme values they found $y_0=0.857d$ and $y_0=0.814d$. Formula (7) with $\beta=0.041$ and $\sigma=0.27$ gives $y_0=0.8414d$, a value which agrees well with the experimental results.

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ON THE METRIZATION OF UNIFORM SPACE

L. W. COHEN AND CASPER GOFFMAN

We call a topological space S metrizable by a group G if there is a distance function for S with values in G which defines an equivalent metric topology in S. It should be noted that for the triangle inequality required of a distance to have meaning, the group G must be ordered, at least partially. In the classical metrization theorem of Chittenden [1],[1] G is the additive group of real numbers. The form of this theorem due to Weil [2] is that a necessary and sufficient condition for a uniform space to be metrizable by the group of real numbers is that it satisfy uniformly the first denumerability axiom of Hausdorff. Kalisch [3], using Weil's theorem that a uniform space is equivalent to a subset of a direct product of metric spaces with real distances, showed that every uniform space is metrizable by a partially ordered group. A related result due to Zelinsky [4] gives necessary and sufficient conditions that a topological field have a valuation in an ordered group.

Here we give necessary and sufficient conditions that a uniform space be metrizable by an ordered abelian group G. We shall assume that the identity in G is not isolated in the order of G, a restriction which excludes the topologically uninteresting discrete spaces. We have shown [5] that the topology of such a group G is determined by a limiting ordinal $\xi^*=\xi^*(G)$ with the property that if $\eta^*<\xi^*$ and ξ_η is a single-valued function on $\eta<\eta^*$ to $\xi<\xi^*$, then

$$(*) \qquad \sup\,[\xi_\eta \mid \eta<\eta^*]<\xi^*;$$

and elements a_ξ, $\xi<\xi^*$, decreasing in G, such that $\inf\,[a_\xi \mid \xi<\xi^*]=\theta$, the identity in G.

THEOREM. *A uniform space S is metrizable by an ordered abelian*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

group G if and only if S has a neighborhood system $\mathcal{U}=[U_\xi(x), \xi<\xi^*, x\in S]$ with the properties

1. $\bigcap_{\xi<\xi^*} U_\xi(x)=[x]$,
2. $\xi_1<\xi_2<\xi^*$ *implies* $U_{\xi_1}(x)\supset U_{\xi_2}(x)$,
3. $\eta<\xi^*$ *implies there is a* $\xi(\eta)$ *such that* $\eta\leqq\xi(\eta)<\xi^*$ *and if* $U_{\xi(\eta)}(x)\cap U_{\xi(\eta)}(y)\neq 0$, *then* $U_{\xi(\eta)}(y)\subset U_\eta(x)$,

4′. $\eta^*<\xi^*$ *implies that* $\bigcap_{\eta<\eta^*} U_{\xi(\eta)}(x)$ *is open, where* ξ^* *is a limiting ordinal with the property* (*).

PROOF. As a preliminary, we remark that, since ω is the first limiting ordinal with the property (*), we may restrict ourselves to the case $\xi^*>\omega$ in view of the classical metrization theorem.

For the necessity: Suppose that $d(x, y)$ is a distance function for S with values in G. The metric topology in S is defined by the spheres $S(x, \xi)=[y \mid d(x, y)<a_\xi]$, $\xi<\xi^*$. Property 1 holds since $\inf a_\xi=\theta$; property 2 holds since the a_ξ decrease on $\xi<\xi^*$; property 3 holds since $d(x, y)$ has the triangle property of a distance function; property 4′ holds since ξ^* is a limiting ordinal with property (*).

For the sufficiency: First we show that S has a neighborhood system $\mathcal{W}=[V_\xi(x)]$, equivalent to $\mathcal{U}$, with properties 1, 2, 3, 4′, and

5. $V_\xi(x)\cap v_\xi(y)\neq 0$ *implies* $V_\xi(x)=V_\xi(y)$.

For $\eta<\xi^*$ we put $\eta_0=\eta$ and $\eta_n=\xi(\eta_{n-1})$, $n<\omega$, where $\xi(\eta_{n-1})$ is the ordinal of 3. Since $\omega<\xi^*$, $\lambda(\eta)=\sup[\eta_n \mid n<\omega]<\xi^*$. We define $\zeta(\eta)=\sup[\lambda(\mu) \mid \mu<\eta+1]$. From 3 and (*) we have $\eta\leqq\xi(\eta)\leqq\lambda(\eta)\leqq\zeta(\eta)<\xi^*$ and $\zeta(\eta_1)\leqq\zeta(\eta_2)$ if $\eta_1<\eta_2$. We define $V_\eta(x)$ as the set of y for which there are $n<\omega$ and $x_1=x, x_2, \cdots, x_n=y\in S$ such that $U_{\zeta(\eta)}(x_k)\cap U_{\zeta(\eta)}(x_{k+1})\neq 0$ for $k=1, \cdots, n-1$.

The equivalence of $\mathcal{W}=[V_\eta(x)]$ and $\mathcal{U}$ is established by showing that $U_{\zeta(\eta)}(x)\subset V_\eta(x)\subset U_\eta(x)$, $\eta<\xi^*$. The first inclusion follows from $y\in U_{\zeta(\eta)}(x)$ and $x_1=x, x_2=y$. For the second, consider $y\in V_\eta(x)$. Then there are $n<\omega$ and $x_1=x, x_2, \cdots, x_n=y\in S$ such that $U_{\zeta(\eta)}(x_k)\cap U_{\zeta(\eta)}(x_{k+1})\neq 0$, $k=1, \cdots, n-1$. From 2 and $\eta_k\leqq\lambda(\eta)\leqq\zeta(\eta)$, we have $U_{\eta_k}(x_k)\cap U_{\eta_k}(x_{k+1})\neq 0$. Since $\eta_k=\xi(\eta_{k-1})$ we have, from 3, $U_{\eta_k}(x_{k+1})\subset U_{\eta_{k-1}}(x_k)$. Hence

$$y=x_n\in U_{\eta_{n-1}}(x_n)\subset U_{\eta_{n-2}}(x_{n-1})\subset\cdots\subset U_{\eta_0}(x_1)=U_\eta(x)$$

and so $V_\eta(x)\subset U_\eta(x)$.

That the system $\mathcal{W}$ has properties 1 and 4′ follows from its equivalence with $\mathcal{U}$. For property 2 consider $\eta'<\eta''$. If $y\in V_{\eta''}(x)$, then for some $x_1=x, x_2, \cdots, x_n=y$, $U_{\zeta(\eta'')}(x_k)\cap U_{\zeta(\eta'')}(x_{k+1})\neq 0$, $k=1, \cdots, n-1$. Since $\mathcal{U}$ has property 2 and $\zeta(\eta')\leqq\zeta(\eta'')$, $U_{\zeta(\eta')}(x_k)\cap U_{\zeta(\eta')}(x_{k+1})\neq 0$. Hence $y\in V_{\eta'}(x)$ and $\mathcal{W}$ has property 2. For property 3 consider

$\eta<\xi^*$ and the ordinal $\xi(\zeta(\eta))$ corresponding to $\zeta(\eta)$ by property 3 for the system $\mathcal{U}$. Now by the inclusions of the equivalence argument, $V_{\xi(\zeta(\eta))}(y)\cap V_{\xi(\zeta(\eta))}(x)\neq 0$ implies $U_{\xi(\zeta(\eta))}(y)\cap U_{\xi(\zeta(\eta))}(x)\neq 0$ and so $V_{\xi(\zeta(\eta))}(y)\subset U_{\xi(\zeta(\eta))}(y)\subset U_{\zeta(\eta)}(x)\subset V_\eta(x)$. Since $\eta\leqq\zeta(\eta)\leqq\xi(\zeta(\eta))$, $\xi(\zeta(\eta))$ serves as the ordinal corresponding to η in property 3 for $\mathcal{W}$.

For property 5 consider $w\in V_\eta(x)\cap V_\eta(y)$ and $z\in V_\eta(y)$. Then there are $x_1'=x, x_2', \cdots, x_p'=w; y_1'=y, y_2', \cdots, y_q'=w; y_1''=y, y_2'', \cdots, y_r''=z$ in S such that $U_{\zeta(\eta)}(u_k)\cap U_{\zeta(\eta)}(u_{k+1})\neq 0$ if $u_k=x_k'$, $k=1, \cdots, p-1$; $u_k=y_k'$, $k=1, \cdots, q-1$; $u_k=y_k''$, $k=1, \cdots, r-1$, respectively. We now have $x_1=x$, $x_2, \cdots, x_{p+q+r-2}=z$ and $U_{\zeta(\eta)}(x_k)\cap U_{\zeta(\eta)}(x_{k+1})\neq 0$, $k=1, \cdots, p+q+r-3$. Hence $z\in V_\eta(x)$ and 5 follows easily.

The ordered group G by which S is metrizable is the additive group of real functions on the ordinals $\xi<\xi^*$. For: Given $x, y\in S$ and $\xi<\xi^*$ we define

$$d_\xi(x, y)=\begin{cases}0, & V_\xi(x)=V_\xi(y),\\ 1, & V_\xi(x)\neq V_\xi(y).\end{cases}$$

For each pair $x, y\in S$ the $d_\xi(x, y)$ are the values of a real function $d(x, y)$ on $\xi<\xi^*$. It is clear that $d(x, y)=d(y, x)\geqq\theta$, the identity in G. Since $\mathcal{W}$ has property 1, $d(x, y)=\theta$ if and only if $x=y$. For the triangle property of $d(x, y)$, consider $x, y, z\in S$ and $\xi<\xi^*$ such that $d_\xi(x, z)=d_\xi(z, y)=0$. Then $V_\xi(x)=V_\xi(z)=V_\xi(y)$ and so $d_\xi(x, y)\leqq d_\xi(x, z)+d_\xi(z, y)$.

Now by the definition of order in G we have $d(x, y)\leqq d(x, z)+d(z, y)$ for all $x, y, z\in S$. Hence $d(x, y)$ is a distance function for S with values in G.

Finally we show that S is metrizable by the group G. Let $a_\xi\in G$, $\xi<\xi^*$, be the functions

$$a_\xi(\eta)=\begin{cases}1, & \eta=\xi,\\ 0, & \eta\neq\xi.\end{cases}$$

For each $\xi<\xi^*$ and $x\in S$, $V_\xi(x)=[y\,|\,d(x, y)<a_\xi]=S(x, \xi)$. For: Consider $y\in V_\xi(x)$. By property 2, $y\in V_\eta(x)$, $\eta\leqq\xi$, and so, by property 5, $d_\eta(x, y)=0$ for $\eta\leqq\xi$. Hence $d(x, y)<a_\xi$. Conversely, if $d(x, y)<a_\xi$, $d_\eta(x, y)=0$ for $\eta\leqq\xi$ and so $V_\xi(x)=V_\xi(y)$. From property 1 it follows that $y\in V_\xi(x)$.

We conclude with 3 remarks: 1. If $\xi^*>\omega$, the metric for S is non-archimedean. 2. The spaces considered in [6] are metrizable by ordered abelian groups. 3. The space of real functions of a real variable with the weak topology is not metrizable by an ordered abelian group.

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A NOTE ON REGULAR AND COMPLETELY REGULAR TOPOLOGICAL SPACES

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For the purposes of this note we shall consider a topological space to be a T-space; that is, the family of open sets for the space is closed under finite intersection and arbitrary union and contains both the void set and the space itself.

We shall be interested in classes of topological spaces having the same points; this common set of points we call the basic point set for the spaces. It is well known that the class of topological spaces $\mathfrak{S}$ for a given basic point set can be partially ordered by means of the inclusion relation for the families of open sets. One such topological space is said to be greater than another if the family of open sets for the former includes the family of open sets for the latter.

With reference to this order relation, $\mathfrak{S}$ is a complete lattice. Indeed, if $\{S_\lambda | \lambda \in \Lambda\}$ is a subset of $\mathfrak{S}$ and $\mathcal{U}_\lambda$ is the family of open sets for S_λ, then $\prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$ is the family of open sets for the greatest lower bound of $\{S_\lambda | \lambda \in \Lambda\}$, and the family $\mathcal{U}$ of open sets for the least upper bound of this subset is given by:

$$\mathcal{U} = \left\{ U \mid \exists \lambda_{\alpha 1}, \lambda_{\alpha 2}, \cdots, \lambda_{\alpha n_\alpha};\ U_{\lambda_{\alpha i}} \in \mathcal{U}_{\lambda_{\alpha i}};\ U = \sum_\alpha U_{\lambda_{\alpha 1}} \cdots U_{\lambda_{\alpha n_\alpha}} \right\}$$

Edwin Hewitt[1] has shown that the least upper bound of a class of regular (completely regular) topological spaces with the same basic point set is itself a regular (completely regular) space provided the original class of spaces is completely ordered. We propose to remove the restriction of complete ordering.

THEOREM. *If* $\mathfrak{S}$, $\{S_\lambda | \lambda \in \Lambda\}$, *is a class of regular (completely regular) topological spaces with the same basic point set, then* S, *the least upper bound of* $\mathfrak{S}$, *is regular (completely regular).*

PROOF. (a) Each member of $\mathfrak{S}$ is regular.

Let s be any point of S, and let U be an open subset of S containing s. Then there exist U_{λ_i} $(i = 1, 2, \cdots, n)$ such that U_{λ_i} is an open subset of S_{λ_i} and

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[1] E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J. vol. 10 (1943) pp 309–333.

$$s \in \prod_{i=1}^{n} U_{\lambda_i} \subset U.$$

Since S_{λ_i} is regular, there exists an open subset of S_{λ_i}, V_{λ_i}, such that $s \in V_{\lambda_i}$ and $V_{\lambda_i}^{C_{\lambda_i}} \subset U_{\lambda_i}$ (where $V_{\lambda_i}^{C_{\lambda_i}}$ denotes the closure of V_{λ_i} in S_{λ_i}). Let V be $\prod_{i=1}^{n} V_{\lambda_i}$. Then $s \in V$ and V is an open subset of S. Finally,

$$V^C = \left(\prod_{i=1}^{n} V_{\lambda_i}\right)^C \subset \prod_{i=1}^{n} V_{\lambda_i}^C \subset \prod_{i=1}^{n} V_{\lambda_i}^{C_{\lambda_i}} \subset \prod_{i=1}^{n} U_{\lambda_i} \subset U.$$

Hence S is a regular space.

(b) Each member of $\mathfrak{S}$ is completely regular.

Let s_0 be any point of S and let U be an open subset of S containing s_0. We pick U_{λ_i} $(i=1, 2, \cdots, n)$ as in (a). Then there exist f_i $(i=1, 2, \cdots, n)$, f_i a continuous real-valued function on S_{λ_i}, such that

$$\begin{aligned} f_i(s_0) &= 0, \\ 0 \leqq f_i(s) &\leqq 1 && (s \in S_{\lambda_i}), \\ f_i(s) &= 1 && (s \in U'_{\lambda_i}). \end{aligned}$$

Let f be the least upper bound of $\{f_i \mid i=1, 2, \cdots, n\}$. Then

$$\begin{aligned} f(s_0) &= 0, \\ 0 \leqq f(s) &\leqq 1 && (s \in S), \\ f(s) &= 1 && (s \in U'). \end{aligned}$$

Since S is stronger than S_{λ_i}, f_i is continuous on S. Hence f is continuous on S. Hence S is completely regular.

A topological space having only the void set and the space itself as open sets is plainly regular (completely regular). Thus any family of topological spaces has a regular (completely regular) lower bound. Considering now the family of regular (completely regular) topological spaces for a given point set as a partially ordered set, we have a partially ordered set with a least element and with least upper bounds for all subsets. Such partially ordered sets are complete lattices. Of course, the greatest lower bound of a family of regular (completely regular) topological spaces in the new lattice need not be the same as the greatest lower bound of the family in the lattice of all topological spaces.

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ON GENERALISING THE NOTION OF FIBRE SPACES TO INCLUDE THE FIBRE BUNDLES[1]

SZE-TSEN HU

1. Introduction. There are two different notions of fibering that have been investigated in the realm of topology. The one is the notion of a fibre bundle, defined by Whitney [11][2] and improved by Steenrod [8]; the other is that of a fibre space, introduced by Hurewicz-Steenrod [6] and generalized by Fox [3]. These two notions do not coincide, as remarked by Steenrod, [8, p. 302]; there are fibre spaces that are not fibre bundles, while it is still unsolved whether every fibre bundle is a fibre space.

According to Fox [3, p. 555], the object of introducing the definition of fibre spaces is to state a *minimum* set of readily verifiable conditions under which the covering homotopy theorem holds. It seems to the author that this set of conditions, given by Hurewicz-Steenrod [6] and Fox [3], is *not* minimum in the sense that it does not naturally apply to the fibre bundles for which the covering homotopy theorem was proved by Steenrod [8, p. 303] in a somewhat weaker form. This weak theorem of covering homotopy of Steenrod works satisfactorily in nearly all the applications where the base space is normal Hausdorff.

In order that the weak theorem of covering homotopy hold, the existence of a unified slicing function in the definition of Hurewicz-Steenrod [6] and in that of Fox [3] is unnecessarily strong. The object of the present paper is to give a generalization of the notion of a fibre space by localizing the slicing function. It will be proved that the generalized class of fibre spaces will include all fibre bundles and that the weak theorem of covering homotopy will still hold.

As mentioned above, there are two forms of the covering homotopy theorem. In the "weak" form, the set being deformed is compact and the homotopy is unrestricted. In the "strong" form, the set is unrestricted but the homotopy is required to be uniform. Since any homotopy of a compact space is uniform, the "strong" theorem implies the "weak" theorem. If the set is completely regular, any uniform homotopy into a compact Hausdorff space can be extended to

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[1] This work was done under Contract N7-onr-434, Task Order No. III, Navy Department, Office of Naval Research.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

a homotopy of its compaction. Therefore, in this case, the "strong" theorem follows from the "weak" theorem. Thus the two theorems are entirely equivalent for compact Hausdorff fibre spaces over Hausdorff base spaces when the set being deformed is completely regular. These relations are studied in §8.[3]

2. **Fiberings.** By a *fibering* F of a topological space X, we understand the collection

$$F = \{X, B, \Omega, \pi, \phi_U\}$$

of the following five entities:

(1) The topological space X, called the *total space*;

(2) A topological space B, called the *base space*;

(3) A system $\Omega = \{U\}$ of open sets U of B which covers B, called the *slicing neighborhoods*;

(4) A continuous map $\pi: X \to B$ of X onto B, called the *projection*;

(5) A system of continuous maps $\{\phi_U\}$ indexed by $U \in \Omega$, called the *slicing functions*, where each ϕ_U is defined on the subset $U \times \pi^{-1}(U)$ of the product space $B \times X$ with images in the total space X in such a way that the following two conditions be satisfied:

$$\pi\phi_U(b, x) = b, \qquad (b \in U, x \in \pi^{-1}(U)); \tag{2.1}$$

$$\phi_U(\pi(x), x) = x, \qquad (x \in \pi^{-1}(U)). \tag{2.2}$$

If a fibering $F = \{X, B, \Omega, \pi, \phi_U\}$ of a given topological space X exists, X will be called a (generalized) *fibre space* over B relative to π and π is said to be a *fibre mapping*. The following statement is an immediate consequence of the above definition and indicates clearly its local character:

(2.3) *A topological space X is a* (generalized) *fibre space over B relative to a continuous onto map $\pi: X \to B$, if and only if, for each point $b \in B$, there exist an open set U of B containing b and a continuous map $\phi_U: U \times \pi^{-1}(U) \to X$ such that the conditions* (2.1) *and* (2.2) *are satisfied.*

By means of the conditions (2.1) and (2.2), the following statement is easy to prove:

(2.4) *Every fibre mapping is open*, that is, it maps open sets onto such.

3. **Fiberings with a unified slicing function.** A fibering $F = \{X, B, \Omega, \pi, \phi_U\}$ is said to *have a unified slicing function*, if, for

[3] The author acknowledges his gratitude to the referee for his suggestions of these relations. In particular, the statements (8.2) and (8.3) were pointed out to the author by the referee.

any two slicing neighborhoods U and V, we always have $\phi_U = \phi_V$ on the intersection of $U \times \pi^{-1}(U)$ and $V \times \pi^{-1}(V)$. For such a fibering F, we may define a *unified slicing function* ϕ on the union W of the subsets $U \times \pi^{-1}(U)$ for all slicing neighborhoods $U \in \Omega$ by taking $\phi = \phi_U$ on each $U \times \pi^{-1}(U)$ and denote the fibering by $F = \{X, B, \Omega, \pi, \phi\}$.

The following theorem is not difficult to prove:

(3.1) *A topological space X is a fibre space over B relative to $\pi: X \to B$ in the sense of Fox* [3, p. 556] *if and only if there exists a fibering $F = \{X, B, \Omega, \pi, \phi\}$ with a unified slicing function.*

We propose to call a fibre space in the sense of Fox (or in the sense of Hurewicz-Steenrod) a *fibre space with a unified slicing function* and to use the term "fibre space" in the generalized sense.

4. **Homotopy type of the fibres.** Let X be a fibre space over B relative to the projection $\pi: X \to B$. For every point $b \in B$, the subset $\pi^{-1}(b)$ of X is called the *fibre* over b. We shall prove the following theorem.

(4.1) *If two points $a, b \in B$ can be connected by a path in B, then $\pi^{-1}(a)$ and $\pi^{-1}(b)$ have the same homotopy type.*

In case the fibre space X has a unified slicing function, this theorem was proved by Hurewicz-Steenrod [6, p. 62]. In their proof, however, an application of the strong theorem of covering homotopy [6, Theorem 1] was made. Since it is still unknown whether the strong theorem of covering homotopy holds for general fibre spaces, we have to give a direct proof as follows.

PROOF. Let $F = \{X, B, \Omega, \pi, \phi_U\}$ be a fibering of X. According to our hypothesis, there is a continuous path $\sigma: I \to B$, where I denotes the closed unit segment $\langle 0, 1 \rangle$ of real numbers, such that $\sigma(0) = a$ and $\sigma(1) = b$. It follows from the compactness of I that there are a finite number of real numbers.

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that, for each closed subsegment $I_i = \langle t_{i-1}, t_i \rangle$ $(i = 1, 2, \cdots, n)$, there is a slicing neighborhood U_i such that $\sigma(I_i) \subset U_i$. Hence, in order to prove (4.1), we can assume without loss of generality that σ maps I into some slicing neighborhood $U \in \Omega$.

Now we assume that $\sigma(I) \subset U$. Define two maps

$$f: \ \pi^{-1}(a) \to \pi^{-1}(b), \qquad g: \ \pi^{-1}(b) \to \pi^{-1}(a)$$

by means of the formula

$$f(x) = \phi_U(b, x), \qquad (x \in \pi^{-1}(a)),$$

$$g(x) = \phi_U(a, x), \qquad (x \in \pi^{-1}(b)).$$

It remains to prove that both gf and fg are homotopic with the identities. Let ι denote the identity map on $\pi^{-1}(a)$. To prove that $gf \simeq \iota$, let us consider the homotopy

$$h_t\colon \quad \pi^{-1}(a) \to \pi^{-1}(a), \qquad (0 \leqq t \leqq 1),$$

defined by taking

$$h_t(x) = \phi_U(a, \phi_U(\sigma(1-t), x)), \qquad (x \in \pi^{-1}(a), 0 \leqq t \leqq 1).$$

Then we have $h_0 = gf$ and $h_1 = \iota$. Hence $gf \simeq \iota$. Similarly, one may prove that fg is homotopic with the identity map on $\pi^{-1}(b)$. This completes the proof.

5. **Fibre bundles.** To prove that the total space of every fibre bundle be a fibre space, we shall recall, in the present section, the definition of a fibre bundle (see [11], [8], [2]).

By a *fibre bundle* Φ, we understand the collection

$$\Phi = \{X, B, \Omega, \pi, D, G, \psi_U\}$$

of the following seven entities:

(1) A topological space X, called the *total space*;

(2) A topological space B, called the *base space*;

(3) A system $\Omega = \{U\}$ of open sets U of B which covers B, called the *coordinate neighborhoods*;

(4) A continuous map $\pi\colon X \to B$ of X onto B, called the *projection*;

(5) A topological space D, called the *director space*;

(6) A topological group G of homeomorphisms of D, called the *reference group*, topologized in such a way that the transformation $G \times D \to D$ defined by the operations of G on D is a continuous map;

(7) A system of homeomorphisms $\{\psi_U\}$ indexed by $U \in \Omega$, called the *coordinate functions*, where each function ψ_U is a homeomorphism of the product space $U \times D$ onto $\pi^{-1}(U) \subset X$ satisfying the condition:

$$\pi\psi_U(b, d) = b, \qquad (b \in B, d \in D). \tag{5.1}$$

Further, these seven entities are supposed to satisfy the following

(5.2) PASTE CONDITION. For a given coordinate neighborhood $U \in \Omega$ and a given point $b \in U$, denote by $\psi_{U,b}$ the homeomorphism of D onto $\pi^{-1}(b)$ defined by $\psi_{U,b}(d) = \psi_U(b, d)$ for each $d \in D$. If b belongs to two coordinate neighborhoods U and V, then $\psi_{V,b}^{-1}\psi_{U,b}$ belongs to G and depends continuously on $b \in U \cap V$.

6. **The fibering associated with a fibre bundle.** Let

$$\Phi = \{X, B, \Omega, \pi, D, G, \psi_U\}$$

be a given fibre bundle. We are going to construct a fibering

$$F = \{X, B, \Omega, \pi, \phi_U\},$$

which will be called the *fibering associated with* Φ.

In order to construct F, it remains only to define the slicing functions ϕ_U. Let $\xi_U: U \times D \to D$ be the projection defined by $\xi_U(b, d) = d$ for each $b \in U$ and each $d \in D$. Then ϕ_U is defined by

$$\phi_U(b, x) = \psi_U(b, \xi_U \psi_U^{-1}(x)), \qquad (b \in U, x \in \pi^{-1}(U)).$$

The conditions (2.1) and (2.2) are verified as follows:

$$\pi\phi_U(b, x) = \pi\psi_U(b, \xi_U\psi_U^{-1}(x)) = b, \qquad (b \in U, x \in \pi^{-1}(U));$$

$$\phi_U(\pi(x), b) = \psi_{U,\pi(x)}\psi_{U,\pi(x)}^{-1}(x) = x, \qquad (x \in \pi^{-1}(U)).$$

The following theorem is an immediate consequence of our associated fibering.

(6.1) *The total space X of any fibre bundle $\Phi = \{X, B, \Omega, \pi, D, G, \psi_U\}$ is a fibre space over B relative to π.*

7. The weak theorem of covering homotopy. With a few almost obvious modifications of the arguments of Steenrod, [8, p. 303], one can prove the weak theorem of covering homotopy for our generalized fibre spaces given as follows:

(7.1) *If X is a fibre space over a normal Hausdorff base space B relative to a projection $\pi: X \to B$, $f: S \to X$ a continuous map of a compact topological space S into X, and $h_t: S \to B$ $(0 \leq t \leq 1)$ a homotopy with $h_0 = \pi f$, then there exists a covering homotopy $f_t: S \to X$ $(0 \leq t \leq 1)$ such that $f_0 = f$ and $\pi f_t = h_t$ for each $0 \leq t \leq 1$. Further, we may assume that, if h_t leaves $s_0 \in S$ fixed so does f_t.*

The following well known homotopy relations in fibre spaces and in fibre bundles are consequences of (7.1):

(7.2) *If X is a fibre space over a normal Hausdorff base space B with pathwise connected fibres, then a necessary and sufficient condition for X to be pathwise connected is the pathwise connectedness of the base space B.*

(7.3) *If X is a pathwise connected fibre space over a normal Hausdorff base space B with pathwise connected fibres and $X_0 = \pi^{-1}(b_0)$ the fibre over $b_0 \in B$, then the projection $\pi: (X, X_0) \to (B, b_0)$ induces (onto) isomorphisms of the homotopy groups*

$$\pi^*: \quad \pi_n(X, X_0) \to \pi_n(B, b_0), \qquad (n \geq 2).$$

The following interesting statement is a consequence of the weak

theorem of covering homotopy and a converse theorem of Fox [4, p. 734]:

(7.4) *Every compact fibre space X over an absolute neighborhood retract B has a unified slicing function.*

8. Uniform homotopy and the strong theorem of covering homotopy. In the present section, the relations between the two forms of the covering homotopy theorem will be clarified.[3]

Let B be a topological space and $\Omega=\{U\}$ be an open covering of B. A homotopy h_t: $S\to B$ $(0\leqq t\leqq 1)$ of a topological space S into B is said to be *uniform with respect to the covering* Ω if there exists a positive real number δ such that, for each $s_0\in S$ and $t_0\in I$, there corresponds an open set $U_0\in\Omega$ such that $h_t(s_0)\in U_0$ for each $t\in I$ satisfying the condition $|t-t_0|\leqq\delta$. h_t is said to be *uniform* if it is uniform with respect to every open covering of B.

The strong theorem of covering homotopy for fibre spaces with a unified slicing function [6, p. 62; 3, p. 557] can be stated as follows:

(8.1) *Let $F=\{X, B, \Omega, \pi, \phi\}$ be a fibering of X with a unified slicing function ϕ. If f: $S\to X$ is a continuous map of a topological space S into X, and h_t: $S\to B$ $(0\leqq t\leqq 1)$ a homotopy which is uniform with respect to Ω and satisfies the relation $h_0=\pi f$, then there exists a covering homotopy f_t: $S\to X$ $(0\leqq t\leqq 1)$ such that $f_0=f$ and $\pi f_t=h_t$ for each $0\leqq t\leqq 1$. Further, we may assume that if h_t leaves $s_0\in S$ fixed, so does f_t.*

To discuss the strong theorem of covering homotopy for general fibre spaces, assume that B be a compact Hausdorff space and S a completely regular space. Let us denote the compaction of S by the usual notation βS (see [9], [1], [10], [7], and [5, p. 53]). We shall prove the following lemma.

(8.2) *Every uniform homotopy h_t: $S\to B$ $(0\leqq t\leqq 1)$ can be extended to homotopy h_t^*: $\beta S\to B$ $(0\leqq t\leqq 1)$.*

PROOF. Imbed B as a subset of the Tychonoff cube Q. The compactness of B implies that B is closed in Q. Since S is dense in βS and every bounded continuous real function defined on S can be continuously extended throughout βS, it follows easily that, for every fixed t, the map h_t: $S\to B$ has a unique extension h_t^*: $\beta S\to B$. Since the given homotopy h_t: $S\to B$ $(0\leqq t\leqq 1)$ is uniform, it is not difficult to prove that the family h_t^*: $\beta S\to B$ $(0\leqq t\leqq 1)$ of maps form a homotopy. This completes the proof.

Now we are going to prove the following strong theorem of covering homotopy for compact Hausdorff fibre spaces.

(8.3) *Let $F=\{X, B, \Omega, \pi, \phi_U\}$ be a fibering of a compact Hausdorff space X over a Hausdorff base space B. If f: $S\to X$ be a continuous map*

of a completely regular space S into X, and $h_t: S \to B$ $(0 \leqq t \leqq 1)$ a uniform homotopy such that $h_0 = \pi f$, then there exists a covering homotopy $f_t: S \to X$ $(0 \leqq t \leqq 1)$ such that $f_0 = f$ and $\pi f_t = h_t$ for each $0 \leqq t \leqq 1$. Further, we may assume that if h_t leaves $s_0 \in S$ fixed so does f_t.

PROOF. Let βS denote the compaction of S. Since X is a compact Hausdorff space, the map $f: S \to X$ has a unique extension $f^*: \beta S \to X$. Since $B = \pi(X)$, the compactness of X implies that of B. According to (8.2), the uniform homotopy $h_t: S \to B$ $(0 \leqq t \leqq 1)$ has a unique extension $h_t^*: \beta S \to B$ $(0 \leqq t \leqq 1)$. It is easy to see that $h_0^* = \pi f$. As a compact Hausdorff space, B is normal. It follows from the weak theorem (7.1) that there exists a covering homotopy $f_t^*: \beta S \to X$ $(0 \leqq t \leqq 1)$ such that $f_0^* = f^*$ and $\pi f_t^* = h_t^*$ for each $0 \leqq t \leqq 1$. Further, we may assume that if h_t leaves $s_0 \in S$ fixed so does f_t^*. The required covering homotopy $f_t: S \to X$ $(0 \leqq t \leqq 1)$ is given by $f_t = f_t^* \mid S$ for each $0 \leqq t \leqq 1$. This completes the proof.

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CONCERNING THE GENERATORS OF HOMOTOPY GROUPS OF A POLYHEDRON

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1. One of the outstanding problems in homotopy theory is to determine the structure of homotopy groups of a given topological space. Even for many simple spaces, very little is known. Given a finite polyhedron P, we also do not know in general whether or not $\pi_n(P)$ is generated by a finite number of elements.

In this note, the following theorem will be proved.

THEOREM 1. *Let P be a finite connected n-dimensional polyhedron with $n \geqq 2$. Let $\pi_t(P)=0$ for all $1<t<n$ but $\pi_n(P)\neq 0$. Then, $\pi_n(P)$ is a group with a finite number of generators if and only if $\pi_1(P)$ is a finite group.*

Let us collect before the proof some notations and elementary facts to be used later. Given a simplicial complex J, by the notations $H_n(J)$ and $Z_n(J)$ will be meant respectively the nth homology group and the group of n-cycles of J, formed by using finite chains with integral coefficients.

Let P be a connected polyhedron and S be the n-sphere $\sum_{i=0}^{n} x_i^2 = 1$ in Euclidean $(n+1)$-space $(n \geqq 2)$. Let ϕ be a mapping: $S \to P$. Then, given any $s, s' \in S$, a path $\lambda_{ss'}$ in S from s to s' gives rise to a path $\phi(\lambda_{ss'})$ in P from $\phi(s)$ to $\phi(s')$.[1] We shall write $\phi_{ss'}$ for $\phi(\lambda_{ss'})$. $\phi_{ss'}$ is determined uniquely by s and s', since S is simply connected.[1] Clearly, $\phi_{ss'}=\phi_{s's}^{-1}$ and $\phi_{ss'}\cdot\phi_{s's''}=\phi_{ss''}$.

Now, let L be a simplicial decomposition of P and p^* be a vertex of L. As usual,[2] we are able to construct a simplicial complex $\bar{L}$ subject to the following conditions:

(i) There is a one-to-one transformation f from the set of vertices of $\bar{L}$ to the set of all ρ_q's where q is a vertex of L, and ρ_q is a path in P from p^* to q.

(ii) Let g be the transformation which carries ρ_q to q and let $\theta=gf$. Then, any $k+1$ mutually distinct vertices $\bar{q}_0, \bar{q}_1, \cdots, \bar{q}_k$ of $\bar{L}$ span a k-simplex of $\bar{L}$ if and only if $\theta(\bar{q}_0), \theta(\bar{q}_1), \cdots, \theta(\bar{q}_k)$ are the distinct vertices of a k-simplex of L and $f(\bar{q}_{j'})$ is the resultant of $f(\bar{q}_j)$ multiplied by the path represented by the oriented segment from $\theta(\bar{q}_j)$ to $\theta(\bar{q}_{j'})$.

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[1] $\lambda_{ss'}$ is a homotopy class rel. 0 and 1 of mappings g: $\langle 0, 1\rangle \to S$ with $g(0)=s$ and $g(1)=s'$. $\phi(\lambda_{ss'})$ is the homotopy class rel. 0 and 1 in P containing the mappings ϕg.

Denote by $\bar{P}$ the polyhedron built of the simplexes of $\bar{L}$. Then, θ gives rise naturally to a simplicial mapping θ: $\bar{P}\to P$. It is known[2] that $\bar{P}$ is a universal covering space of P with covering mapping θ. We have

$$\pi_1(\bar{P}) = 0. \tag{1}$$

Let us assume further that

$$\pi_i(P) = 0 \text{ for a certain } n \geqq 2 \text{ and all } 1 < i < n. \tag{2}$$

Then, by the covering mapping θ,

$$\pi_i(\bar{P}) = 0 \text{ for all } 1 < i < n, \qquad \pi_n(\bar{P}) \approx \pi_n(P). \tag{3}$$

This together with (1) and an isomorphism of Hurewicz[3] gives

$$\pi_n(P) \approx H_n(\bar{L}). \tag{4}$$

Thus, under the condition (2), we have: (i) If P is a finite polyhedron and $\pi_1(P)$ is a finite group, then $\bar{L}$ is a finite complex and $H_n(\bar{L})$ as well as $\pi_n(P)$ is a group with a finite number of generators; (ii) if dim $P=n$, then dim $\bar{P}=n$ and $H_n(\bar{L})\approx Z_n(\bar{L})$ and hence it follows from (4) that

$$\pi_n(P) \approx Z_n(\bar{L}). \tag{5}$$

Proof of Theorem 1. The proof of the sufficiency of the theorem has been given above. The necessity follows from the following more general statement (Δ).

Let P be a connected n-dimensional polyhedron (not necessarily finite) which fulfils the condition (2). *If $\pi_1(P)$ is an infinite group and $\pi_n(P) \neq 0$, then $\pi_n(P)$ contains an infinite number of independent elements.*[4]

We now prove the statement (Δ). First, taking a vertex p^* of a simplicial decomposition L of P as base point, construct the universal covering space $\bar{P}$ of P as above. Let the notations $\bar{L}, f, \theta, S$ retain their usage here. Let s^* be a point of S and let η be a nonzero element of $\pi_n(P, p^*)$. Decompose S into a simplicial complex K so that we have

[2] H. Seifert and W. Threfall, *Lehrbuch der Topologie*, Leipzig, Teubner, 1934, pp. 189–194.

[3] A mapping h: $S\to P$, where S is an oriented n-sphere, and P an arcwise connected topological space, determines a singular cycle $\psi(h)=(h, S)$ in P. If $\pi_i(P)=0$ for all $0<i<n$, ψ gives rise to an isomorphism of $\pi_n(P)$ onto the nth singular homology group of P with integral coefficients. This isomorphism will be referred to as an isomorphism of Hurewicz.

[4] We mean that $\pi_n(P)$ contains an infinite set A such that, for each positive integer k, k mutually distinct elements of A are independent.

a simplicial mapping ϕ: $(S, s^*) \to (P, p^*)$ which represents η. Denote by B the set of all elements of $\pi_1(P, p^*)$ of the form $\phi_{s^*r} \cdot \phi_{r's^*}$ where r and r' are vertices of K with $\phi(r) = \phi(r')$. Clearly, B is a finite set.

Suppose that $\pi_1(P, p^*)$ consists of an infinite number of elements. Let us define a sequence $\{\xi_i\}$ of elements of $\pi_1(P, p^*)$, taking ξ_1 arbitrarily first, and then, by mathematical induction on supposing that $\xi_1, \xi_2, \cdots, \xi_i$ have already been defined, taking ξ_{i+1} as an arbitrary element of $\pi_1(P, p^*)$ which does not belong to the finite set $\bigcup_{j=1}^{i} \xi_j \cdot B$.[5] We see that the sequence $\{\xi_i\}$ so obtained possesses the property that, for any $i \neq i'$, $\xi_i^{-1} \cdot \xi_{i'} \notin B$.

To each ξ_i, we are able to associate a transformation ψ_i from the vertices of K to the vertices of $\bar{L}$ subject to the condition:

$$f\psi_i(r) = \xi_i \cdot \phi_{s^*r} \text{ for any vertex } r \text{ of } K. \tag{6}$$

Since $\xi_i \cdot \phi_{s^*r}(\xi_i \cdot \phi_{s^*r}) \cdot \phi_{rr'}$ for any vertices r and r' of K, by the construction of $\bar{P}$, ψ_i gives rise naturally to a simplicial mapping ψ_i: $S \to \bar{P}$. Clearly $\phi = \theta\psi_i$. Let us denote by α the fundamental n-cycle of K. Then, $\psi_i(\alpha) \in Z_n(\bar{L})$.

We see that every $\psi_i(\alpha) \neq 0$. For, if $\psi_i(\alpha) = 0$, then, since P fulfills (2), we have from (1), (3), and an isomorphism of Hurewicz[3] that $\psi_i \cong 0$ in $\bar{P}$; it follows that $\phi \simeq 0$ in P, which contradicts that $\eta \neq 0$. We see also that for any two distinct i and i', the absolute complexes $|\psi_i(\alpha)|$ and $|\psi_{i'}(\alpha)| \subset \bar{L}$ are disjoint. For, if not, there are vertices r and r' of K such that $\psi_i(r) = \psi_{i'}(r')$ and hence by (6) $\xi_i \cdot \phi_{s^*r} = \xi_{i'} \cdot \phi_{s^*r'}$ which contradicts that $\xi_{i'}^{-1} \cdot \xi_i \notin B$. Moreover, the cycles $\psi_i(\alpha)$'s are infinite in number. It follows therefore that $Z_n(\bar{L})$ contains an infinite number of independent elements.

Since P fullfils the condition (2) and dim $P = n$, we have therefore the isomorphism (5). This completes the proof of the statement (Δ) and hence that of Theorem 1.

2. Let n be an integer not less than 2. Given any two pathwise connected topological spaces $P \subset P'$ and p^*, a point of P, we shall always use ι to denote the injection homomorphism: $\pi_n(P, p^*) \to \pi_n(P', p^*)$. Now, let us consider three pathwise connected topological spaces $Q, R \subset P$. Let s^* be a point of $Q \cap R$, and let ι^* be the homomorphism: $\pi_n(Q, s^*) + \pi_n(R, s^*)$ [direct sum] $\to \pi_n(P, s^*)$ defined by taking $\iota^*(\eta) = \iota(\xi) + \iota(\zeta)$ where $\xi \in \pi_n(Q, s^*)$, $\zeta \in \pi_n(R, s^*)$ and $\eta = \xi + \zeta$. It follows from some arguments given by G. W. Whitehead that:[6]

[5] $\xi_i \cdot B$ is the set of all such element $\xi_i \cdot \zeta$ of $\pi_1(P, p^*)$ with $\zeta \in B$.

[6] G. W. Whitehead, *A generalization of the Hopf invariant*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 188–190.

LEMMA 1. *If $Q \cup R = P$ and $Q \cap R = (s^*)$, ι^* is an into-isomorphism.*

THEOREM 2. *Let P be a connected polyhedron, and P', the polyhedron obtained by identifying two distinct points p_1 and p_2 of P to a single point. Then, if $\pi_n(P) \neq 0$ for $n \geqq 2$, $\pi_n(P')$ cannot be a group with a finite number of generators.*

To prove the theorem, we construct a polyhedron

$$Q = \bigcup_{i=0,\pm1,\pm2,\cdots} Q_i$$

subject to the conditions: (i) For each i, there is a homeomorphism f_i of Q_i onto P; (ii) Q_i and Q_{i+1} are in contact at the point $q_i = f_i^{-1}(p_1) = f_{i+1}^{-1}(p_2)$ and have no other common points than q_i; (iii) $Q_i \cap Q_{i+j} = 0$ if $j > 1$. Clearly, Q is a covering space of P'. We have

$$\pi_n(Q_i) \approx \pi_n(P) \text{ for every } i, \tag{7}$$

and since $n \geqq 2$, we have

$$\pi_n(Q) \approx \pi_n(P'). \tag{8}$$

Given any pair of integers $m \leqq m'$, denote by $Q_{mm'}$ the subpolyhedron $\bigcup_{m \leqq i \leqq m'} Q_i$ of Q. Making use of Lemma 1 by changing successively the base point to form a homotopy group, we shall have a subgroup of $\pi_n(Q_{mm'})$ which is isomorphic to the direct sum $\pi_n(Q_m) + \pi_n(Q_{m+1}) + \cdots + \pi_n(Q_{m'})$ and which the injection homomorphism $\iota: \pi_n(Q_{mm'}) \to \pi_n(Q)$ maps isomorphically into $\pi_n(Q)$. Therefore, by (7), given any positive integer k, $\pi_n(Q)$ contains a subgroup isomorphic to the direct sum of k groups $\approx \pi_n(P)$. This together with (8) and some arguments in group theory proves Theorem 2.

COROLLARY. *Under the same hypotheses of Theorem 2, if $n \geqq 2$ and $\pi_n(P') \neq 0$, then $\pi_n(P')$ cannot be a group with a finite number of generators.*

To prove this, as in the proof of Theorem 2, we construct the covering space Q of P'. If $n \geqq 2$ and $\pi_n(P') \neq 0$, then we have an essential mapping of the n-sphere into some $Q_{mm'}$, that is, $\pi_n(Q_{mm'}) \neq 0$ for some $Q_{mm'}$. Clearly, Q is also a covering space of a certain polyhedron $Q'_{mm'}$ obtained by identifying two distinct points of $Q_{mm'}$. It follows therefore from Theorem 2 that $\pi_n(P') \approx \pi_n(Q) \approx \pi_n(Q'_{mm'})$ is not a group with a finite number of generators.

THEOREM 3. *Let P be a connected polyhedron, and P' the polyhedron obtained by identifying two distinct points p_1 and p_2 of P. Then, if $n \geqq 2$ and $\pi_t(P) = 0$ for all $1 < t < n$, $\pi_t(P')$ is the weak direct sum of an infinite countable number of groups each of which is isomorphic to $\pi_n(P)$.*

We precede the proof of the theorem by two lemmas. Given two polyhedra $P \subset P^*$, and two points $p_1 \neq p_2$ of P, we shall say that P^* is a segmental extent of P through the points p_1 and p_2, if there is a simplicial decomposition L^* of P^* such that p_1 and p_2 span a ground 1-simplex of L^* and P is built of all simplexes of L^* other than the simplex $|p_1p_2|$, that is, $P = \overline{L^* - |p_1p_2|}$.

LEMMA 2. *Let P^* be a segmental extent of a polyhedron P through the points $p_1 \neq p_2$ of P, and let P' be the polyhedron obtained by identifying the points p_1 and p_2. Then P^* and P' have the same homotopy type.*

This can be easily obtained by some standard procedure in homotopy theory. The detailed proof will be omitted here.

LEMMA 3. *Let P^* be a segmental extent of a connected polyhedron P through the points $p_1 \neq p_2$ of P. Then*: (i) *The injection homomorphism $\iota: \pi_1(P, p_1) \to \pi_1(P^*, p_1)$ is an into-isomorphism*; (ii) *$\pi_1(P^*, p_1)$ is isomorphic to the free product of $\pi_1(P, p_1)$ and an infinite cyclic group.*

To prove this, by the definition of segmental extents, there is a simplicial decomposition L^* of P^* such that p_1 and p_2 span a ground 1-simplex of L^* and $P = \overline{L^* - |p_1p_2|}$. Since P is connected, there is a simple arc A contained in the 1-skeleton of P and joining p_1 and p_2. Clearly, $S^1 = |p_1p_2| \cup A$ is homeomorphic to a circle.

Denote also by ι the injection homomorphism: $\pi_1(S^1, p) \to \pi_1(P^*, p_1)$. Given an arbitrary element $\xi = \xi_1\xi_2 \cdots \xi_t$ of the free product $F = \pi_1(P, p_1) \circ \pi_1(S^1, p_2)$ where $\xi_i \in \pi_1(P, p_1)$ or $\pi_1(S^1, p_1)$, put $\iota^*(\xi) = \iota(\xi_1)\iota(\xi_2) \cdots \iota(\xi_t)$. Then, since $\pi_1(A, p_1) = 1$, ι^* establishes an onto-isomorphism: $F \to \pi_1(P^*, p_1)$.[7] (ii) follows now therefore at once, since $\pi_1(S^1, p_1)$ is cyclic infinite. (i) follows from the fact that the homomorphism $\iota: \pi_1(P, p_1) \to \pi_1(P^*, p_1)$ agrees with ι^* over the subgroup $\pi_1(P, p_1)$ of F.

PROOF OF THEOREM 3. Without loss of generality, we may assume that P is a polyhedron contained in a sufficiently higher-dimensional Euclidean space or in the Hilbert space so that it has a segmental extent P^* through p_1 and p_2. Let L^* be a simplicial decomposition of P^* such that p_1 and p_2 span a ground 1-simplex of L^* and $P = \overline{L^* - |p_1p_2|}$. By Lemma 2 it suffices for us to prove that $\pi_n(P^*)$ is the weak direct sum of an infinite countable number of groups $\approx \pi_n(P)$. Let us construct as in §1 a simplicial complex $\bar{L}^*$ such that the polyhedron $\bar{P}^*$ built of the simplexes of $\bar{L}^*$ is a universal covering space of P^* with a simplicial covering mapping θ.

Denote by $\bar{P}_i$ an arbitrary component of $\theta^{-1}(P)$. $\bar{P}_i$ is then a cover-

[7] H. Seifert and W. Threfall, loc. cit. pp. 177–179.

ing space of P with covering mapping $\theta|\bar{P}_i$. Let f be a mapping: $S^1 \to \bar{P}_i$ where S^1 is a circle. Then $f \simeq 0$ in $\bar{P}^*$ and hence $\theta f \simeq 0$ in P^*. It follows from Lemma 3(i) that $\theta f \simeq 0$ in P and hence by the covering mapping $\theta|\bar{P}_i$ that $f \simeq 0$ in $\bar{P}_i$. Thus, $\bar{P}_i$ is also a universal covering space of P. Next, we see that every $\bar{P}_i$ is a polyhedron composed of the simplexes of a subcomplex $\bar{L}_i$ of $\bar{L}^*$. Clearly, a simplex of $\bar{L}^*$ is of dimension 1 if it is not in any $\bar{L}_i$. We have therefore that $H_t(\bar{L}^*)$ is the weak direct sum of the $H_t(\bar{L}_i)$'s for $t \geqq 2$. By a theorem of Hurewicz,[3] we obtain easily that $\pi_n(P^*)$ is the weak direct sum of the $\pi_n(\bar{P}_i)$'s.

Since $\pi_n(\bar{P}_i) \approx \pi_n(P)$, it remains only to show that the $\bar{P}_i$'s are countably infinite. But this follows from the fact that the components of $\theta^{-1}(P)$ are in one-one correspondence with the collection of left cosets of $i(\pi_1(P))$ in $\pi_1(P^*)$. By Lemma 3 this collection is countably infinite.

Repeating the use of Theorem 3 by taking $n = 2, 3, 4, \cdots$, successively, we get the following corollary.

COROLLARY. *Let P be a connected polyhedron, and P' the polyhedron obtained by identifying two distinct points of P. Then P' is aspherical if and only if P is aspherical.*

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UNSTABLE HOMEOMORPHISMS

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1. **Definitions.** Let f be a homeomorphism of a metric space X (with metric d) onto itself. f will be called *unstable* on X provided there is a number $\delta(f, X) > 0$ (called an instability constant) such that corresponding to each pair of distinct points x, y of X, there is an integer $n(x, y)$ for which

$$d(f^n(x), f^n(y)) > \delta.$$

As an example of an unstable homeomorphism let X be the Euclidean plane. Select a point O of X. Define $f(O) = O$ and if $O \neq p \in X$, define $f(p) = q$ where q is the point of the ray Op, with end point O, such that $d(O, q) = 2d(O, p)$. Now f is unstable, with δ arbitrary, since if $d(x, y) = r$,

$$d(f^n(x), f^n(y)) = 2^n \cdot r.$$

Let M be a metric space and let $\phi(M) = M$ be a homeomorphism. $O(m) \equiv \sum_{-\infty}^{+\infty} \phi^i(m)$ is called the *orbit* of $m \in M$ under ϕ, $O_-(m) \equiv \sum_{-\infty}^{-1} \phi^i(m)$ is called the *negative semi-orbit* of $m \in M$ under ϕ, and $O_+(m) \equiv \sum_{0}^{+\infty} \phi^i(m)$ is called the *positive* semi-orbit of $m \in M$ under ϕ. The point p is said to be an *α-limit point* (*ω-limit point*) of the orbit O if there exists a point $x \in O$ and a sequence of integers $n_1 > n_2 > \cdots$ ($n_1 < n_2 < \cdots$) such that $\lim \phi^{n_i}(x) = p$. The set of all α-limit points and the set of all ω-limit point of an orbit O are closed sets which are invariant under ϕ (cf. [3, §2]).[1]

$O(m)$ will be called *positively* (*negatively*) *asymptotic* to a set $A \subset M$ under ϕ if, given $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that $n > N$ ($n < N$) implies $\phi^n(m) \in V_\epsilon(A)$, the ϵ-neighborhood of A. Let x and y be distinct points of M. $O(x)$ and $O(y)$ will be called positively (negatively) asymptotic if, given $\epsilon > 0$, there exists an integer N such that $n > N$ ($n < N$) implies

$$d(\phi^n(x), \phi^n(y)) < \epsilon.$$

2. **Instability and asymptoticity.** A space is said to be dense-in-itself if every point of the space is a limit point of points of the space. Unless otherwise qualified X, in this section, shall denote a compact metric space that is dense-in-itself and $f(X) = X$ shall be an unstable

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[1] Numbers in brackets refer to the references cited at the end of the paper.

homeomorphism with instability constant δ.

(2.1) THEOREM. *X contains a pair of points whose orbits are asymptotic in at least one sense.*

PROOF. Consider the product space $X^2 \equiv X \times X$ and the homeomorphism $g(X^2) = X^2$ defined as $g(x_1 \times x_2) = f(x_1) \times f(x_2)$, $x_1, x_2 \in X$. If $z_1 = x_1 \times y_1$ and $z_2 = x_2 \times y_2$ are elements of X^2, let

$$\rho(z_1, z_2) = d(x_1, x_2) + d(y_1, y_2).$$

With this definition, X^2 becomes a metric space.[2]

If D denotes all elements of X^2 of the form $x \times x$, $x \in X$, then D is compact and invariant under g. Let U be the δ-neighborhood of D in X^2. If $z = x \times y$ is a point of U, then $\rho(z, D) < \delta$, that is, $d(x, p) + d(y, p) < \delta$ for some $p \in X$. However, $d(x, y) \leqq d(x, p) + d(y, p) < \delta$, hence $d(x, y) < \delta$. Thus, since f is unstable on X, the only points of X^2 whose orbits (under g) lie in U are points of D.

Let $N(D)$ be a neighborhood of D in X^2 such that its closure, $\overline{N}(D)$, is a subset of U and let $\{z_i\}$ be a sequence of points of $X^2 - D$ such that $z_i \to z \in D$. Since $\overline{N}(D) \subset U$ and f is unstable on X, no orbit $O(z_i)$, $i = 1, 2, \cdots$, can be entirely in $\overline{N}(D)$. Thus for each i there is an integer m_i such that

$$g^{m_i}(z_i) \notin \overline{N}(D).$$

For each i we can assume that m_i has the smallest absolute value for any integer with the stated property. There are either an infinite number of the collection $\{m_i\}$ that are positive or an infinite number that are negative, or both. Suppose that there are an infinite number of positive elements of the sequence and let $\{m_i\}$ denote this class. In case there are an infinite number of negative elements of the original sequence, the proof is similar. The collection can be considered as ordered so that the elements increase with their subscripts. There exists a point $s \in X^2 - N(D)$ such that for some subcollection of $\{m_i\}$

$$g^{m_i}(z_i) \to s.$$

Hereafter, $\{m_i\}$ is to denote this subsequence of the initial sequence.

The negative semi-orbit of s lies in $\overline{N}(D)$ since

$$g^{m_i - m}(z_i) = g^{-m}(g^{m_i}(z_i)) \to g^{-m}(s)$$

and for i sufficiently large and $m > 0$,

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[2] $g(X^2) = X^2$ inherits instability from f with the same instability constant.

$$g^{m_i - m}(z_i) \in \overline{N}(D).$$

Let O denote the orbit of s. Since $\overline{N}(D)$ is closed, O_α, the set of α-limit points of O, is in $\overline{N}(D)$. Moreover O_α is invariant and if O is not negatively asymptotic to the set D, there exists a point α_0 in O_α but not in D such that

$$O(\alpha_0) \subset O_\alpha \subset \overline{N}(D) \subset U.$$

As was remarked, an orbit remains in U only if it is the orbit of a point of D; thus we conclude that O is negatively asymptotic to D.

If $s = x \times y$, then for $\epsilon > 0$ there exists an N such that for $n < N$, $\rho(g^n(s), D) < \epsilon$. This implies $d(f^n(x), f^n(y)) < \epsilon$ and the proof of the theorem is complete.

An unstable homeomorphism followed by an unstable homeomorphism is not necessarily unstable. However, under some conditions a combination of homeomorphisms is always unstable. In order to continue the investigation of asymptotic orbits we digress to prove the following theorem.

(2.2) THEOREM. *Let X be compact and metric and let $f(X) = X$ be an unstable homeomorphism. For any integer $m \neq 0$, $f^m(X) = X$ is unstable.*

PROOF. We first recall that if X is compact and metric and $M(X) = X$ is a homeomorphism, then given $\xi > 0$ there exists $\eta > 0$ such that $d(x, y) > \xi$ implies $d(M(x), M(y)) > \eta$. Now, let $\phi(X) \equiv f^m(X)$ and consider the homeomorphisms $f^i(X)$, $i = \pm 1, \pm 2, \cdots, \pm m$. By the remark above, there exist numbers $\eta_i > 0$ $(i = \pm 1, \pm 2, \cdots, \pm m)$ such that $d(x, y) > \delta$ implies $d(f^i(x), f^i(y)) > \eta_i$, for all $x, y \in X$. We shall show that $\min \{\eta_i\}$ is an instability constant for $\phi(X)$. If x and y are distinct points of X, there exists an n such that

$$d(f^n(x), f^n(y)) > \delta.$$

There exists an integer r such that $0 < |rm - n| \leqq m$ and if i is taken as $rm - n$, then

$$f^i(f^n(s)) = f^{rm}(s) = \phi^r(s), \qquad s \in X.$$

Since $d(f^n(x), f^n(y)) > \delta$, by the remark above $d(\phi^r(x), \phi^r(y)) > \min \{\eta_i\}$ and hence $\phi(X)$ is unstable with instability constant $\min \{\eta_i\}$.

(2.3) LEMMA. *If $x \in X$ is a fixed point of $f(X) = X$, then there exists a point of X whose orbit is asymptotic to x in at least one sense.*

PROOF. Let U be the δ-neighborhood of x and let $\{x_i\}$ be a sequence

of points of $X-(x)$ for which $\lim x_i=x$. Let $N\subset U$ be the closure of a neighborhood of x. No matter how large i is taken, $O(x_i)$ intersects $X-N$ since f is unstable. As in the proof of (2.1), let m_i be the smallest integer in absolute value for which

$$f^{m_i}(x_i) \notin N.$$

Either an infinite number of the m_i's are positive or an infinite number are negative, or both. Assume that an infinite number are positive. If this is not the case, the proof is similar. We can select a sequence of these integers that go to infinity with their subscripts and such that the points $f^{m_i}(x_i)$ converge to a point $y\in X-N$. The negative semi-orbit of y is asymptotic to x by arguments similar to those used in the proof of (2.1).

(2.4) THEOREM. *If $x\in X$ is periodic under f, then there exists a point of X whose orbit is asymptotic to $O(x)$ in at least one sense.*

PROOF. If x is of period m, x is a fixed point under $f^m(X)$. By (2.2), $f^m(X)$ is unstable and by the previous lemma there is a point $y\in X$ such that $O(y)$ is asymptotic (with respect to $f^m(X)$) to x in at least one sense. Assume $O(y)$ is asymptotic to x under $f^m(X)$ in the positive sense (the other possibility can be similarly treated). Then for $\epsilon>0$ given, there exists an integer K_0 such that $i>K_0$ implies

$$d(f^{im}(y), f^{im}(x)) < \epsilon.$$

Moreover, for a given integer j there exists an integer K_j such that $i>K_j$ implies

$$d(f^{im+j}(y), f^{im+j}(x)) < \epsilon.$$

If $K=\max(K_0, K_1, \cdots, K_{m-1})$, then for $i>K$,

$$d(f^{im+j}(y), f^{im+j}(x)) < \epsilon, \qquad j = 0, 1, 2, \cdots, m-1.$$

Thus there exists an integer N (namely $(K+1)m$) such that $n>N$ implies

$$d(f^n(y), f^n(x)) < \epsilon.$$

This completes the proof of the theorem.

Let X be a metric space and let $f(X)=X$ be a homeomorphism. The point $x\in X$ will be called an *unstable point* under f provided there exists a number $\delta(x)>0$ such that if y is any point of X distinct from x, then for some integer $n(y)$,

$$d(f^n(x), f^n(y)) > \delta.$$

It is easily seen that if x is unstable under f, then each point of $O(x)$ is unstable under f (for the same δ). The following statements follow from the proofs of (2.2), (2.3), and (2.4).

(i) *If X is compact and metric and $x \in X$ is an unstable point under the homeomorphism $f(X) = X$, then for any integer $m \neq 0$, x is unstable under $f^m(X) = X$.*

(ii) *If x is a limit point of the compact metric space X and is unstable and periodic under the homeomorphism $f(X) = X$, then there is a point of X whose orbit is asymptotic to $O(x)$ in at least one sense.*

The homeomorphism of the unit interval onto itself defined by $x \to x^3$ is an example of a transformation under which some, but not all, points are unstable.

3. Additional results and examples.

(3.1) THEOREM. *Let X be compact and metric and let $f(X) = X$ be an unstable homeomorphism. If K is any positive integer, the points of X of period K are finite in number. Thus the periodic points of X form a countable set.*

PROOF. If X contains an infinity of fixed points, they have a limit point which is a fixed point. At this fixed point instability is violated. If X contains an infinity of points of period K, under $f^K(X) = X$ there are an infinity of fixed points, contrary to the instability of f^K. Thus the points of period K are finite and the theorem is proved.

A compact dense-in-itself space has the potency of the continuum [5, p. 68]. Thus, by (3.1), there is no pointwise periodic unstable homeomorphism on such a space.

A transformation $\phi(M) = M$ is said to be *regular* on M, a metric space, if given a real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that if x and y are points of X for which $d(x, y) < \delta$, then $d(\phi^n(x), \phi^n(y)) < \epsilon$ for all integers n (that is, ϕ has equi-continuous powers). A point x of a metric space X is said to be *almost periodic* under the homeomorphism $\phi(X) = X$ if given any real number $\epsilon > 0$, there exists a relatively dense (successive differences bounded) sequence of integers $\{n_i\}$ unbounded above and below such that $d(\phi^{n_i}(x), x) < \epsilon$ for all integers i. The transformation ϕ is said to be almost periodic if, for each $\epsilon > 0$, every point of X is almost periodic under ϕ for the same sequence $\{n_i\}$. Clearly, instability is a strong contradiction of regularity on a space that is dense-in-itself. It is known [2] that if ϕ is almost periodic on a compact metric space, then ϕ is regular. Thus an unstable transformation on a compact, metric space that is dense-in-itself cannot be almost periodic. However, in

the examples cited below there is an unstable homeomorphism that is pointwise almost periodic on such a space.

Examples of orbits asymptotic under unstable homeomorphisms can be found in the symbol spaces extensively investigated by Morse and Hedlund.[3] If $\mathfrak{E}$ is the space based on two symbols, then $\mathfrak{E}$ is compact, metric, and dense-in-itself. Hedlund [3] has pointed out a pair of doubly asymptotic orbits in $\mathfrak{E}$. Another example consists of the closure, M, of the orbit T [4, p. 832] due to Morse. This subset of $\mathfrak{E}$ is a minimal set on which the transformation is pointwise almost periodic. In M there is an orbit T' [4, p. 833] positively asymptotic to T, an orbit S negatively asymptotic to T, and still a fourth orbit negatively asymptotic to T' and positively asymptotic to S. Other examples can be constructed in $\mathfrak{E}$.

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[3] $\mathfrak{E}$ and the unstable homeomorphism of $\mathfrak{E}$ onto itself are defined, for example, in [1], [3], and [4].

THE RELATION BETWEEN THE CLASS NUMBER AND THE DISTRIBUTION OF PRIMES

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We shall present here a very simple proof of a connection of the distribution of primes and the class number of a quadratic imaginary algebraic number field. All constants in the following are easily computable. No other method is available to compute the class number with exact constants. The following method, however, is based on a conditional hypothesis.

Let $d \equiv 3 \pmod 4$ and a prime; $k(d)$ the class number of the field $R((-d)^{1/2})$ where R is the rational numbers; $\pi(z; d, t)$ the number of primes not greater than z and congruent to $t \pmod d$.

THEOREM. *If* $\pi(z; d, t) < Az/(d-1) \log z$ *for any* z *with* $d < z < e^{c(d)^{1/2}/r}$ *and for all* t *which are nonresidues* $\pmod d$; *where*

$$A < 2, \quad c = (1/25)(1 - A/2),$$

then $h(d) > r$ *for* $d > N = N(A, r)$.

PROOF. By hypothesis the number of primes not greater than z and quadratic residues of k must be not less than $(1 - A/2)(z/\log z)$, as the maximum number of primes which are nonresidues is $(A/2)(z/\log z)$.

If[1] $h(d) = s \leqq r$, these primes must be represented by one of the following s reduced forms:

$$a_i x^2 + b_i xy + c_i y^2 = \frac{1}{4a_i}\left((2a_i x + b_i y)^2 + d y^2\right)$$

where $0 < a_i \leqq (d/3)^{1/2}$, $|b_i| \leqq (d/3)^{1/2}$ for $i = 1, 2, \cdots, s$.

We shall now compute an upper bound on the number of all numbers not greater than z for $d < z < e^{c(d)^{1/2}/r}$ and represented by one of the above s forms.

Now

$$(2a_i x + b_i y)^2 + d y^2 \leqq 4a_i z$$

has at most $2(4a_i z/d)^{1/2} + 1$ different positive and negative integer solutions for y, as clearly by above $|y| \leqq (4a_i z/d)^{1/2}$. Hence, for a fixed y,

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[1] We shall assume here that all the elementary facts on the representation of numbers by binary quadratic forms are known. E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1.

$$|2a_ix| \leqq (4a_iz)^{1/2} + |b_iy|$$
$$\leqq (4a_iz)^{1/2}\left(1 + \frac{|b_i|}{d^{1/2}}\right).$$

Therefore, there are at most $2(z/a_i)^{1/2}(1+|b|/d^{1/2})+1$ different solutions for x.

Therefore, at most

$$\left(4\left(\frac{a_iz}{d}\right)^{1/2}+1\right)\left\{2\left(\frac{z}{a_i}\right)^{1/2}\left(1+\frac{|b|}{d^{1/2}}\right)+1\right\}$$
$$< \left(5\left(\frac{a_iz}{d}\right)^{1/2}\right)\left(5\left(\frac{z}{a_i}\right)^{1/2}\right) = 25\frac{z}{d^{1/2}}$$

(utilizing our bounds on a_i and b_i) numbers not greater than z can be represented by any one form.

Therefore, at most $25(z/d^{1/2})s$ numbers not greater than z can be represented by all s forms. Therefore,

$$25\frac{sz}{d^{1/2}} \geqq \left(1-\frac{A}{2}\right)\frac{z}{\log z},$$
$$\log z \geqq \frac{1}{25s}\left(1-\frac{A}{2}\right)d^{1/2}$$
$$\geqq \frac{1}{25r}\left(1-\frac{A}{2}\right)d^{1/2}$$
$$\geqq \frac{c}{r}\,d^{1/2}$$

which is a contradiction to the definition of z.

It is interesting to note that Viggo Brun has shown $\pi(z;\ d,\ l) < Az/d \log z$ for $z \geqq d^{1+\epsilon}$, but the A is about 6. Selberg's recent work yields an improvement in the value A, to A slightly greater than 2.

The above method works for any positive d. When $d \equiv 3 \pmod 4$ and a prime, the problem of the greatest d such that $h(d)=1$ has not yet been settled.

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THE SOLUTION OF A UNILATERAL DIRECT PRODUCT MATRIX EQUATION

JAMES H. BELL

Introduction. In the study of families of solutions of the unilateral matrix equation, over a field $\mathfrak{F}$ of characteristic zero, the matrix equation $\sum_{m=0}^{s} A_m \cdot \times X^m = 0$ occurred. Necessary and sufficient conditions were obtained for the solution of this equation.[1] The more general equation $\sum_{m=0}^{s} A_m \cdot \times (K_m X^m) = 0$ is considered in this paper. The solution of this general equation is reduced to the solution of a system of simultaneous unilateral matrix equations, which are equal in number to the number of linearly independent (over $\mathfrak{F}$) matrices in the set $A_s, A_{s-1}, \cdots, A_0$. Also, it is shown how, under certain conditions, the equation may possibly be solved using a method paralleling the method of M. H. Ingraham [1][2] for the solution of the unilateral matrix equation. In this connection, an interesting remainder theorem and a divisor theorem are obtained.

In this paper the matrices involved in the equations will have elements in $\mathfrak{F}$. For the sake of brevity $A \cdot \times B = (Ab_{ij})$ will be written $A \times B$. Let λ be a scalar indeterminate. A matrix with elements in the polynomial ring $\mathfrak{F}[\lambda]$ will be termed a *λ-matrix*. A square *λ-matrix* which has an inverse which is also a λ-matrix is called *unimodular*. If $TA = B$ where T, A, and B are λ-matrices and T is unimodular, then A is said to be a *left associate* of B.

Every square λ-matrix is the left associate of a unique λ-matrix of the following form: Every element below the main diagonal is zero. If the main diagonal element is nonzero, it is a monic polynomial and the other elements in its column are of lesser degree in λ. In the case that the main diagonal element is zero, then all of the elements in its row are also zero. This matrix will be referred to as the *canonical triangular form or c.t.f.*[3]

1. The general equation. In this section X will be considered to be an $n \times n$ matrix, A_m $(m = 0, 1, \cdots, s)$ will be an $r \times p$ matrix,

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[1] J. H. Bell, *Families of solutions of the unilateral matrix equation*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 151–159.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] The canonical triangular form is obtained in the same manner as the Hermite normal form [2], except that the operations are carried out on the columns in reverse order.

and K_m will be $t\times n$. If $M=A^{r\times p}\times B^{t\times n}=(m_{ij})^{rt\times pn}$, the matrix $[M]_{ij}$ will be defined to be the matrix $(m_{(k-1)r+i,(l-1)p+j})$, $k=1,2,\cdots,t$; $l=1,2,\cdots,n$. It follows that $[M]_{ij}=a_{ij}B$, $[M]_{ij}+[N]_{ij}=[M+N]_{ij}$, and $[M]_{ij}\equiv 0$ for all i and j, if and only if $M\equiv 0$.

THEOREM 1. *The equation* $\sum_{m=0}^{s} A_m\times(K_mX^m)=0$ *over* $\mathfrak{F}$, *where* $A_m=(a_{m,ij})^{r\times p}$, *has a solution if and only if the unilateral matrix equations* $\sum_{m=0}^{s} a_{m,ij}K_mX^m=0$ $(i=1,2,\cdots,r;\ j=1,2,\cdots,p)$ *have a common solution. These simultaneous equations may be reduced in number to an equivalent set equal in number to the number of linearly independent (over* $\mathfrak{F}$*) matrices in the set* $A_s, A_{s-1},\cdots,A_0$.

Let $Q=\sum_{m=0}^{s} A_m\times(K_mX^m)$, then $[Q]_{ij}=\sum_{m=0}^{s} a_{m,ij}K_mX^m$. Therefore, $Q=0$ if and only if $[Q]_{ij}=0$ for every i and j. That is, X is a solution of $Q=0$ if and only if X is a solution of the simultaneous set of equations $\sum_{m=0}^{s} a_{m,ij}K_mX^m=0$.

Since each equation is of degree s, and there are rp equations, it may be possible to pick out a basic set of linearly independent equations. Any solution of the basic equations will be a solution of the whole set. The scalar coefficients (the $a_{m,ij}$) of any equation will be linear combinations of the coefficients of the basic equations. Form the matrix $(a_{s,ij}, a_{s-1,ij},\cdots,a_{0,ij})$ $(i=1,2,\cdots,r;j=1,2,\cdots,p)$. The rank of this matrix will equal the number of linearly independent equations and will also equal the number of linearly independent columns in the matrix. Each column is one of the matrices A_m considered as a vector in rp space. Hence the number of linearly independent equations equals the number of linearly independent matrices among $A_s, A_{s-1},\cdots,A_0$.

THEOREM 2. *If the matrix* K_m $(m=0,1,\cdots,s)$ *is square and of order* n, *the equation* $\sum_{m=0}^{s} A_m\times(K_mX^m)=0$ *will have a solution only if the* rp *polynomials formed by taking the determinant of* $\sum_{m=0}^{s} a_{m,ij}K_m\lambda^m$, *for every* i *and* j, *have a common divisor of degree* n.

If X is a solution of $\sum_{m=0}^{s} a_{m,ij}K_mX^m=0$, then the characteristic polynomial of X must divide the determinant of $\sum_{m=0}^{s} a_{m,ij}K_m\lambda^m$ [1].

COROLLARY. *The equation* $\sum_{m=0}^{s} A_m\times X^m=0$ *has a solution if and only if the polynomials* $\sum_{m=0}^{s} a_{m,ij}\lambda^m$ *have a common divisor* $\sum_{m=0}^{g} d_m\lambda^m$ *which is not a unit, and is such that* $\sum_{m=0}^{g} d_mX^m=0$ *has a solution.*

2. **An alternative method.** If all of the matrices are square although the order of A_m may be different from the order of X, the solution of the equation may be carried out by a method paralleling Ingraham's method for the solution of the unilateral equation [1].

Let $\Lambda = \lambda I^n \times^n$ and form the matrix $\sum_{m=0}^{s} A_m \times (K_m \Lambda^m)$. This matrix will be a λ-matrix. It is a well known theorem that $(A \times B)(C \times D) = (AC \times BD)$, if A and C, and B and D, are of the same orders.

THEOREM 3. *If A and B are two square λ-matrices, not necessarily of the same order, and if A^H and B^H are the canonical triangular forms of A and B respectively, then $(A \times B)^H = A^H \times B^H$. If $T_1 A = A^H$ and $T_2 B = B^H$, then $(T_1 \times T_2)(A \times B) = (A \times B)^H$.*

Suppose A and B are λ-matrices and the c.t.f.'s of A and B are A^H and B^H respectively where $T_1 A = A^H$ and $T_2 B = B^H$. It follows that $(T_1 \times T_2)(A \times B) = A^H \times B^H$. An inspection of $A^H \times B^H$ will show that it is in canonical triangular form. Therefore, $A^H \times B^H = (A \times B)^H$, since $(T_1 \times T_2)$ is unimodular.

THEOREM 4. *If $Q = \sum_{m=0}^{s} A_m \times (K_m \Lambda^m)$, then $Q = S(I \times (\Lambda - X)) + \sum_{m=0}^{s} A_m \times (K_m X^m)$, where I is of the same order r as A_m, and S is an $rn \times rn$ λ-matrix.*

The proof of Theorem 4 is by induction on s since

$$A_1 \times (K_1 \Lambda) + A_0 \times K_0 = (A_1 \times K_1)(I \times (\Lambda - X)) + A_1 \times (K_1 X) + A_0 \times K_0$$

and

$$\sum_{m=0}^{s} A_m \times (K_m \Lambda^m) = (A_s \times (K_s \Lambda^{s-1}))(I \times (\Lambda - X)) + A_s \times (K_s X \Lambda^{s-1}) + \sum_{m=0}^{s-1} A_m \times (K_m \Lambda^m).$$

COROLLARY. *The matrix X is a solution of $\sum_{m=0}^{s} A_m \times (K_m X^m) = 0$ if and only if $\sum_{m=0}^{s} A_m \times (K_m \Lambda^m) = S(I \times (\Lambda - X))$.*

This corollary follows directly from Theorem 4.

Since the matrix X is a solution of $\sum_{m=0}^{s} A_m \times (K_m X^m) = 0$ if and only if $Q = \sum_{m=0}^{s} A_m \times (K_m \Lambda^m) = S(I \times (\Lambda - X))$, then X is a solution if and only if $Q^H = R(I \times (\Lambda - X))$. Let $A = (\Lambda - X)^H$, then $TA = \Lambda - X$ where T is unimodular. Also, $(I \times (\Lambda - X))^H = I \times A$. Therefore, X is a solution if and only if there exists a matrix A in canonical triangular form such that A is the left associate of $\Lambda - X$, that is, $TA = \Lambda - X$, and $Q^H = R(I \times T^{-1})(I \times T)(I \times (\Lambda - X)) = P(I \times A)$.

The problem of solving the original equation is thus reduced to that of factoring Q^H so as to obtain a right divisor which is of the

form $(I\times A)$, where A is the c.t.f. of a matrix $\Lambda - X$ and X has elements in $\mathfrak{F}$.

A matrix A is the c.t.f. of a matrix $\Lambda - X$ if: The degree of $\prod_{m=1}^{t} a_{mm}$ is less than or equal to i, the degree of $\prod_{m=1}^{n} a_{mm}$ is equal to n, and if $A = \sum_{m=0}^{p} B_m \Lambda^m$ the matrix $W_1 = \|B_p, B_{p-1}, \cdots, B_1\|$ is of rank n. These conditions are necessary and sufficient [3].

Let $Q^H = (q_{ij})$. Since Q^H and $I\times A$ are triangular matrices, it follows that P is a triangular matrix and

$$q_{(i-1)r+t,(i-1)r+t} = p_{(i-1)r+t,(i-1)r+t} a_{ii} \qquad (t = 1, 2, \cdots, r). \tag{1}$$

Thus a_{ii} $(i=1, 2, \cdots, n)$ must be picked to satisfy (1) and the conditions listed above.

The other elements of the matrix A are obtained in the following manner: In any column, say the jth: $a_{jj}, a_{j-1,j}, \cdots, a_{1,j}$ are obtained in the order listed, and the columns are obtained in the order $j=1, 2, \cdots, n$. The matrix $I\times A$ is composed of diagonal matrices A_{ij} of order r, and $A_{ij} = a_{ij} I^{r\times r}$.

Assume that all of the elements up to a_{li} have been found, then all p_{kj} $(j \leqq (i-1)r;\ k > lr,\ j=(i-1)r+1,\ (i-1)r+2, \cdots, ir)$ are determined. The following system of equations determines a_{li}:

$$p_{(l-1)r+t,(l-1)r+j} = 0, \qquad t > j;$$

and

$$\sum_{m=l}^{i} p_{(l-1)r+t,(m-1)r+j} a_{mi} = q_{(l-1)r+t,(i-1)r+j}, \qquad 1 \leqq t \leqq j \leqq r.$$

That is,

$$p_{(l-1)r+t,(l-1)r+j} a_{li} \equiv q_{(l-1)r+t,(i-1)r+j} - \sum_{m=l+1}^{i-1} p_{(l-1)r+t,(m-1)r+j} a_{mi} \bmod a_{ii}. \tag{2}$$

Therefore a_{li} must be a solution of $n(n+1)/2$ linear congruences. These congruences may have no solution, a unique solution, or a family of solutions. If at any stage it is impossible to solve the congruences (2), then there is no factorization A having the chosen main diagonal elements.

It is interesting to note that if $l=i$,

$$q_{(i-1)r+t,(i-1)r+j} \equiv 0 \bmod a_{ii}, \qquad t \leqq j.$$

The above method gives an algorithm for the complete solution of

the equation $\sum_{m=0}^{s} A_m \cdot \times K_m X^m = 0$, if the matrices involved are square.

Results similar to those in this paper may be obtained, in some cases more readily, in the consideration of $\sum_{m=0}^{s} (X^m K_m) \cdot \times A_m = 0$, $\sum_{m=0}^{s} A_m \times \cdot (K_m X^m) = 0$, and $\sum_{m=0}^{s} (X^m K_m) \times \cdot A_m = 0$, and so on, where $A \times \cdot B = (a_{ij}B)$.

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AVERAGES OF CHARACTER SUMS

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Suppose that χ is a primitive residue character[1] modulo k, $k>1$, and that for y non-negative, $S(y) = \sum_{0 \le l \le y} \chi(l)$. It is important [see, for example, 11] in the analytic theory of numbers to have as much information as possible about the sums $S(y)$, in particular about their maximum order of magnitude; it is known (cf. [13; 14; 8]), for example, that $S(y) < k^{1/2} \log k$, but unknown whether or not $M(\chi) = o(k^{1/2} \log k)$ as k tends to infinity, where $M(\chi)$ is the maximum of $|S(1)|, \cdots, |S(k-1)|$. Hua [4; 5; 6] has shown that it is often helpful to consider the averages $n^{-1} \sum_{m=0}^{n} S(m)$. In this paper we consider some further developments of this idea.

1. **Preliminaries.** We recall [7, pp. 483–486, 492–494] that if χ is a primitive residue character mod k and if $\tau(\chi) = \sum_{n=1}^{k} \chi(n) e^{2\pi i n/k}$, then $|\tau(\chi)| = k^{1/2}$ and

$$\sum_{n=1}^{k} \chi(n) e^{2\pi i m n/k} = \overline{\chi}(m)\tau(\chi) \tag{1}$$

for any integer m, $\overline{\chi}$ being the complex conjugate of χ.

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[1] For the basic facts about residue characters see [7, pp. 401–414, 478–494]. Numbers in brackets refer to the bibliography.

The function $S(kx)$ is of bounded variation and period unity, and as such has an everywhere convergent Fourier series. It may be deduced from (1) (cf. [13, pp. 23–24] and [8, pp. 81–82]) that this Fourier series is

$$A(\chi) - \frac{\tau(\chi)}{2\pi i} \sum_{m \neq 0} \frac{\bar{\chi}(m)}{m} e^{-2\pi i m x}, \tag{2}$$

where m runs over the positive and negative integers and

$$A(\chi) = \int_0^1 S(kx)dx = \frac{1}{k} \sum_{m=0}^{k-1} S(m). \tag{3}$$

Thus if we define a function $S^*(y)$ as $S(y)$ if y is not an integer and as $S(y) - \chi(y)/2$ if y is an integer, we have

$$S^*(kx) = \begin{cases} A(\chi) + \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin 2\pi n x & \text{if } \chi(-1) = 1 \\ A(\chi) - \dfrac{\tau(\chi)}{\pi i} \displaystyle\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \cos 2\pi n x & \text{if } \chi(-1) = -1 \end{cases} \tag{4}$$

for all non-negative x.

2. **Another proof of Paley's Ω-result for** $S(n)$. If we put $x = 0$ in (4) we get a formula for the arithmetic mean $A(\chi)$ of $S(0), \cdots, S(k-1)$, namely

$$A(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ \dfrac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) & \text{if } \chi(-1) = -1, \end{cases} \tag{5}$$

where $L(s, \bar{\chi}) = \sum_{n=1}^{\infty} \bar{\chi}(n) n^{-s}$ for $R(s) > 0$. The second half of formula (5) can of course be proved by the method used in [9] to prove the second part of Satz 217 (of which the second half of (5) is a generalization), but some use of Fourier series is essential.[2]

The second part of (5) enables us, when $\chi(-1) = -1$, to use information about the order of magnitude of $L(1, \bar{\chi})$ to get information about the order of magnitude of $A(\chi)$. For example, a slight change in the argument of [2] shows that for real primitive χ with $\chi(-1) = -1$ we have

$$\limsup_{k \to \infty} \frac{L(1, \chi)}{\log \log k} \geqq e^{\gamma}, \tag{6}$$

[2] Naturally (5) can be proved in a roundabout way by showing that both sides are equal to $L(0, \chi)$. The argument indicated here essentially stems from Minkowski (cf. [13, p. 26]).

γ being Euler's constant. (The lim sup is unambiguous, since for each k there is at most one real primitive χ with $\chi(-1)=-1$.) Now $\tau(\chi)=ik^{1/2}$ for real primitive χ with $\chi(-1)=-1$ [9, Satz 215] and thus (6) gives the following result.

THEOREM 1. *If χ runs through all real primitive characters with $\chi(-1)=-1$ in order of increasing size of the modulus k of χ and if $A(\chi)$ is as defined in* (3), *then*

$$\limsup_{k\to\infty}\frac{A(\chi)}{k^{1/2}\log\log k}\geqq\frac{e^{\gamma}}{\pi}.$$

This theorem is a slightly stronger form of Paley's result (see [12; 10; 1]) that if χ is a primitive character modulo k, then $M(\chi)=\Omega(k^{1/2}\log\log k)$, where $M(\chi)$ is the maximum of $|S(0)|,\cdots,|S(k-1)|$; for our theorem shows that this Ω-result is true even for the arithmetic mean of $S(0),\cdots,S(k-1)$.

We remark in passing that Parseval's formula applied to (2) or (4) gives

$$\frac{1}{k}\sum_{m=0}^{k-1}|S(m)-A(\chi)|^2=\int_0^1|S(kx)-A(\chi)|^2dx$$

$$=\frac{k}{2\pi^2}\sum_{n=1}^{\infty}{}'\frac{1}{n^2}=\frac{k}{12}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right),$$

where the $'$ indicates summation over the positive integers relatively prime to k. Alternatively we have, in view of (5),

$$\frac{1}{k}\sum_{m=0}^{k-1}|S(m)|^2=\begin{cases}\dfrac{k}{2\pi^2}\displaystyle\sum_{n=1}^{\infty}{}'\frac{1}{n^2} & \text{if } \chi(-1)=1\\[2ex] \dfrac{k}{\pi^2}\left(\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}{}'\frac{1}{n^2}+|L(1,\chi)|^2\right) & \text{if } \chi(-1)=-1.\end{cases}$$

3. **Estimation of certain partial averages of $S(n)$.** We saw above that if χ is a primitive character with $\chi(-1)=1$, then the arithmetic mean of the numbers $S(0),\cdots,S(k-1)$ is zero. Hence in this case we should expect a fairly good estimate of the arithmetic mean of $S(0),\cdots,S(n-1)$ for $n<k$. And in fact Hua [4; 5; 6] has proved that we have

$$(7)\qquad\left|\frac{1}{n}\sum_{m=0}^{n-1}S(m)\right|\leqq\frac{1}{2}\left(k^{1/2}-\frac{n}{k^{1/2}}\right)\quad\text{for}\quad 0\leqq n\leqq k,\ \chi(-1)=1.$$

If χ is a primitive character with $\chi(-1)=-1$, such a neat result

cannot be expected, since the arithmetic mean of $S(0), \cdots, S(k-1)$ is not zero. However we prove the following result, which is valuable for n not too small relative to k, especially if $|L(1, \chi)|$ is large.

THEOREM 2. *If χ is a primitive character modulo k with $\chi(-1) = -1$ and if $\alpha k \leqq n \leqq k$, where α is a number between 0 and 1, then*

$$\left| \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) \right| < \left(2.1 + \frac{\pi}{4} \log \frac{1}{\alpha} \right) k^{1/2}.$$

PROOF. Put $q = [k/2]$, $\rho = e^{2\pi i/k}$. Then by (1) we have

$$\begin{aligned}
\tau(\bar{\chi}) S(m) &= \sum_{l=0}^{m} \chi(l) \tau(\bar{\chi}) = \sum_{l=0}^{m} \sum_{h=-q}^{q} \bar{\chi}(h) \rho^{hl} \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \sum_{l=0}^{m} (\rho^{hl} - \rho^{-hl}) \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(m+1)} - 1 - \rho^{h} + \rho^{-hm}}{\rho^{h} - 1} \\
&= \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(m+1)} + \rho^{-hm}}{\rho^{h} - 1} \\
&\quad + i \sum_{h=1}^{q} \bar{\chi}(h) \cot \frac{\pi h}{k}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\tau(\bar{\chi})}{n} \sum_{m=0}^{n-1} S(m) &= \frac{1}{n} \sum_{h=1}^{q} \bar{\chi}(h) \frac{\rho^{h(n+1)} - \rho^{-h(n-1)}}{(\rho^{h} - 1)^2} \\
&\quad + i \sum_{h=1}^{q} \bar{\chi}(h) \cot \frac{\pi h}{k} \\
&= \frac{1}{2in} \sum_{h=1}^{q} \bar{\chi}(h) \frac{\sin (2\pi hn/k)}{\sin^2 (\pi h/k)} \\
&\quad + i \sum_{h=1}^{q} \bar{\chi}(h) \cot \frac{\pi h}{k}.
\end{aligned}$$

Now $\tau(\chi)\tau(\bar{\chi}) = -\tau(\chi)\overline{\tau(\chi)} = -k$ for $\chi(-1) = -1$ and thus

$$\begin{aligned}
&\tau(\bar{\chi}) \left\{ \frac{1}{n} \sum_{m=0}^{n-1} S(m) - \frac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) \right\} \\
&\qquad = \frac{\tau(\bar{\chi})}{n} \sum_{m=0}^{n-1} S(m) - \frac{ik}{\pi} L(1, \bar{\chi})
\end{aligned}$$

$$(8)\qquad = \frac{1}{2in}\sum_{h=1}^{q}\bar{\chi}(h)\frac{\sin(2\pi hn/k)}{\sin^2(\pi h/k)} + i\sum_{h=1}^{q}\bar{\chi}(h)\left(\cot\frac{\pi h}{k}-\frac{1}{\pi h/k}\right) - \frac{ik}{\pi}\sum_{h=q+1}^{\infty}\frac{\bar{\chi}(h)}{h} = R_1+R_2+R_3,$$

say. If we put $S_1(h)=\sum_{n=q+1}^{h}\bar{\chi}(n)$ and use the fact that $|S_1(h)|<2^{-k}$, we have

$$(9)\qquad |R_2| = \left|\frac{k}{\pi}\sum_{h=q+1}^{\infty}\frac{S_1(h)}{h(h+1)}\right| \leqq \frac{2^{-1}k^2}{\pi}\sum_{h=q+1}^{\infty}\frac{1}{h(h+1)} = \frac{2^{-1}k^2}{\pi(q+1)} \leqq \frac{k}{\pi}.$$

Using the fact that $0<x^{-1}-\cot x\leqq 2/\pi$ for $0<x\leqq\pi/2$, we find

$$(10)\qquad |R_2| \leqq \frac{2}{\pi}q \leqq \frac{1}{\pi}k.$$

We divide the summation in R_1 into two parts, namely $1\leqq h\leqq[\alpha^{-1}+1]$ and $[\alpha^{-1}+2]\leqq h\leqq q$. In the first part we use $\sin(2\pi hn/k)\leqq 2\pi hn/k$, in the second part we use $\sin(2\pi hn/k)\leqq 1$, and in both parts we use $\sin(\pi h/k)\geqq 2h/k$. Thus

$$(11)\qquad \begin{aligned} |R_1| &\leqq \frac{1}{2n}\sum_{h=1}^{[\alpha^{-1}+1]}\frac{2\pi hn/k}{4h^2/k^2}+\frac{1}{2n}\sum_{h=[\alpha^{-1}+2]}^{q}\frac{1}{4h^2/k^2}\\ &\leqq \frac{\pi}{4}k\sum_{h=1}^{[\alpha^{-1}+1]}\frac{1}{h}+\frac{k^2}{8n}\sum_{h=[\alpha^{-1}+2]}^{\infty}\frac{1}{h^2}\\ &\leqq \frac{\pi}{4}k\left(\frac{3}{2}+\log\frac{1}{\alpha}\right)+\frac{k^2}{8n}\alpha\\ &\leqq \left(\frac{3\pi}{8}+\frac{\pi}{4}\log\frac{1}{\alpha}+\frac{1}{8}\right)k. \end{aligned}$$

Combining (8), (9), (10), and (11), we get our theorem.

4. A theorem of Davenport. Davenport [3] has proved that if s is a fixed complex number with $0<\sigma=R(s)<1$, then for any primitive χ we have $|L(s,\chi)|\leqq Ck^{(1-\sigma)/2}$ where C is a constant depending on s. In this section we show how Hua's inequality (7) gives a very simple proof of this result in the case $\chi(-1)=1$, with a specific value of the

constant. Unfortunately our Theorem 2 is too weak to give Davenport's result for the case $\chi(-1)=-1$.

THEOREM 3. *If χ is a primitive character with $\chi(-1)=1$ and if $0<\sigma=R(s)<1$, then*

$$|L(s,\chi)| \leqq \frac{|s(s+1)|}{\sigma(1-\sigma)}\, k^{(1-\sigma)/2}.$$

PROOF. Put $T(n)=\sum_{m=0}^{n} S(m)$. Then we get by two partial summations

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\sum_{n=1}^{\infty}S(n)s\int_0^1\frac{dy}{(n+y)^{s+1}}$$

$$=\sum_{n=1}^{\infty}T(n)s(s+1)\int_0^1\int_0^1\frac{dxdy}{(n+x+y)^{s+2}}.$$

For $n\leqq[k^{1/2}]$ clearly $|T(n)|\leqq\sum_{m=0}^{n}m\leqq n^2$, while for $n\geqq[k^{1/2}+1]$ we have by (7)

$$|T(n)|\leqq 2^{-1}k^{1/2}\{n+1-(n+1)^2/k\}\leqq 2^{-1}k^{1/2}n.$$

Thus

$$|L(s,\chi)|\leqq|s(s+1)|\sum_{n=1}^{\infty}\frac{|T(n)|}{n^{\sigma+2}}$$

$$=|s(s+1)|\left\{\sum_{n=1}^{[k^{1/2}]}\frac{1}{n^{\sigma}}+\sum_{n=[k^{1/2}]+1}^{\infty}\frac{2^{-1}k^{1/2}}{n^{\sigma+1}}\right\}$$

$$\leqq|s(s+1)|\left\{1+\int_1^{[k^{1/2}]}\frac{dx}{x^{\sigma}}+2^{-1}k^{1/2}\int_{[k^{1/2}]}^{\infty}\frac{dx}{x^{\sigma+1}}\right\}$$

$$\leqq|s(s+1)|\left\{1+\frac{k^{(1-\sigma)/2}-1}{1-\sigma}+\frac{2^{-1}k^{1/2}}{\sigma[k^{1/2}]^{\sigma}}\right\}$$

$$\leqq|s(s+1)|\left\{\frac{k^{(1-\sigma)/2}}{1-\sigma}+\frac{k^{(1-\sigma)/2}}{\sigma}\right\}=\frac{|s(s+1)|}{\sigma(1-\sigma)}\,k^{(1-\sigma)/2},$$

which proves our theorem.

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INSTITUTE FOR ADVANCED STUDY

CONCERNING A CERTAIN SET OF ARRANGEMENTS

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1. Introduction, definitions, and notations. Let m and k be two positive integers, with $k \leqq m$. A set S of arrangements (permutations) of the m elements

$$1, 2, \cdots, m-1, m$$

will be called k-suitable, if for every k-subset of the m elements, say

$$a_1, a_2, \cdots, a_k$$

there exists an arrangement in S in which a_k follows all the a_i with $i<k$. Such sets S surely exist; for example, any set of m arrangements whose terminal elements are 1, 2, $\cdots$, m, respectively, will obviously be k-suitable for any $k \leqq m$.

The smallest cardinal number N such that there exists a set of N arrangements which is k-suitable for m will be denoted by $N(m, k)$; this is therefore a well defined positive integer for any m and k, $k \leqq m$.

In any collection of arrangements of the first m positive integers, the set of the terminal elements of the several arrangements will be called the end-elements, while all the other elements will then be referred to as mid-elements.

We shall hereafter refer to "suitable" sets of arrangements (omitting the numerical prefix) if the value of k is clearly indicated by the context, or if the reference applies to all values of k.

In this paper we are primarily concerned with the evaluation of $N(m, k)$. The evaluation is not complete; the main result is given in Theorem I. In the last sections we use the results obtained to formulate an answer to a problem in connection with partially ordered sets.

Throughout this paper, M will denote the set of the first m positive integers. Small letters, insofar as they represent numbers, will represent positive integers.

2. Some preliminary lemmas.

LEMMA 1. *If* $1 \leqq k_1 \leqq k_2 \leqq m_1 \leqq m_2$, *then* $N(m_1, k_1) \leqq N(m_1, k_2) \leqq N(m_2, k_2)$.

LEMMA 2. $k \leqq N(m, k) \leqq m$.

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LEMMA 3. $N(m, 1)=1$, $N(m, 2)=2$, $N(m, m)=m$.

LEMMA 4. *There exists a suitable set of arrangements in which the terminal elements of the several arrangements are all distinct. Moreover, if the set of end-elements for this set of arrangements is denoted by*

$$a_1, a_2, \cdots, a_s,$$

then the arrangement which terminates with a_i $(i=1, 2, \cdots, s)$ *begins with an (arbitrary) arrangement of the* a_j, $j \neq i$.

The first three statements follow readily from the definitions. To prove Lemma 4, let

$$A: \quad A_1, A_2, A_3, \cdots, A_s$$

be a suitable set of s arrangements. By Lemma 2, we may suppose that $s \leqq m$. Let a_1 be the terminal element of A_1, and let A_i', $i \neq 1$, be the arrangement obtainable from A_i by transposing in it (if necessary) the element a_1 to the beginning, leaving the other elements undisturbed. It is easy to see that the set of arrangements

$$A': \quad A_1, A_2', A_3', \cdots, A_s'$$

is still suitable. Now let a_2 be the terminal element of A_2'; a_2 is clearly different from a_1. Let $A_1'', A_2'', A_4'', \cdots, A_s''$ be the arrangements obtained from $A_1, A_2', A_4', \cdots, A_s'$ by transposing in each one the element a_2 to the beginning, leaving the other elements undisturbed. Once again, the set of arrangements

$$A'': \quad A_1'', A_2', A_3'', \cdots, A_s''$$

will remain suitable. Proceeding in the indicated manner, we shall finally get (after s steps at most) a set of arrangements of the sort described in the lemma.

LEMMA 5. *For any natural* m, *the expression*

$$\frac{m}{x} + x - 1$$

is monotone decreasing as x *increases from* 1 *to* $m^{1/2}$.

To prove this, we need only to examine the graph of the hyperbola $xy = m + x^2 - x$.

3. We shall now prove the following theorem.

THEOREM I. *For any integral* $m \geqq 4$, *let* j *be any integer from* 2 *to*

$[m^{1/2}]$ *inclusive,*[1] *and let* k *be any integer which satisfies the inequality*

$$\text{(A):} \qquad \left[\frac{m+j^2-j}{j}\right] \leqq k < \left[\frac{m+(j-1)^2-j+1}{j-1}\right].$$

Then $N(m, k) = m-j+1$.[2]

PROOF. For a given integral j between 2 and $[m^{1/2}]$ inclusive, the smallest value of k defined by the inequality (A) is

$$k_1 = \left[\frac{m}{j}\right] + j - 1,$$

and the largest value is

$$k_2 = \left[\frac{m}{j-1}\right] + j - 3$$

(with the alternatives that either no value of k is defined, or that $k_2 = k_1$). Suppose now that we have already established the following facts:

(a) For k_2, one can construct a set of $m-j+1$ arrangements which is suitable; while

(b) no set of $m-j$ arrangements can be k_1-suitable.

Then, by (a) and (b),

$$N(m, k_2) \leqq m - j + 1, \qquad N(m, k_1) > m - j,$$

and the truth of our theorem will follow immediately from Lemma 1.

As to (a), consider $m-j+1$ arrangements which terminate respectively with the elements $1, 2, \cdots, m-j, m-j+1$, and for which the arrangement which terminates with s, $1 \leqq s \leqq m-j+1$, begins with an arbitrary arrangement of the other $m-j$ end-elements. The remaining elements $m-j+2, m-j+3, \cdots, m-1, m$, which constitute the set of mid-elements here, and whose number is clearly $j-1$, will therefore appear in each arrangement in the $j-1$ places immediately preceding the terminal element. The ordering of the mid-elements among themselves will be as follows. Separate the set of arrangements into $j-1$ mutually exclusive subsets, so that each subset contains either $[(m-j+1)/(j-1)]$ or $1+[(m-j+1)/(j-1)]$ arrangements, the separation being otherwise arbitrary. Now let the first subset have the element $m-j+2$ as the penultimate element in each of its arrangements, and let each of the remaining $j-2$ mid-

[1] As usual, $[x]$ will denote the largest integer not exceeding x.

[2] The case of $m=1$, 2, or 3 or $j=1$ is taken care of by Lemma 3.

elements immediately precede the element $m-j+2$ in at least one arrangement of this subset. This is possible, since

$$\left[\frac{m-j+1}{j-1}\right] = \left[\frac{m}{j-1}\right] - 1 \geqq \left[\frac{m-1}{m^{1/2}-1}\right] - 1$$
$$\geqq m^{1/2} - 1 \geqq j - 2.$$

We proceed similarly with the arrangements of each of the other subsets; thus, in the arrangements of the last subset the element m will occupy the penultimate position, and for each of the other mid-elements there will be at least one arrangement of this subset in which this element will immediately precede m.

This set of $m-j+1$ arrangements is certainly k_2-suitable. Indeed, consider any k_2-subset of M, such as

(C): $$a_1, a_2, \cdots, a_t$$

(where we put $k_2 = t$ for typographical reasons). If a_t if an end-element, the suitability is obvious. If a_t is a mid-element, consider the subset of arrangements in which a_t is the penultimate element, and suppose that *not all* of the end-elements of this subset are members of (C). In this case, again, there will surely be an arrangement wherein a_t follows all the other a's. Finally, suppose that *all* the end-elements of the subset, in which a_t is the penultimate element, belong to (C). The number of all other elements of (C) is thus at most

$$k_2 - 1 - \left(\left[\frac{m}{j-1}\right] - 1\right)$$
$$= \left[\frac{m}{j-1}\right] + j - 3 - 1 - \left[\frac{m}{j-1}\right] + 1 = j - 3.$$

Since the total number of subsets is $j-1$, it follows that there will be at least one subset such that neither the end-elements nor the penultimate element of the arrangements in this subset will be members of (C). Hence, an arrangement in this subset will exhibit a_t as following all the other a's.

The proof of (a) is now complete.

Proof of (b). Suppose that the contrary is true: let S be a set of $m-j$ arrangements of M which is k_1-suitable. We suppose further that S is of the sort described in Lemma 4 of §2, so that the penultimate element of any arrangement in S is a mid-element. Since the total number of mid-elements is j, there will be one of them, call it a_j, which occurs as the penultimate element in at most

$$t \equiv \left[\frac{m-j}{j}\right]$$

arrangements of the set S. Denote the several end-elements of these arrangements by

$$b_1, b_2, \cdots, b_q \qquad (q \leqq t)$$

and let

$$a_1, a_2, \cdots, a_{j-1}$$

denote all the mid-elements other than a_j. Now

$$q + j \leqq t + j = \left[\frac{m-j}{j}\right] + j = \left[\frac{m}{j}\right] + j - 1 = k_1,$$

so that

$$\{c\};\ b_1, b_2, \cdots, b_q;\ a_1, a_2, \cdots, a_j$$

represents a k_1-subset ($\{c\}$ denotes the set of elements other than the a's and b's; it is empty if $t=q$). But it is now clear that in any arrangement of S where a_j is not followed by a b, it will be followed by a_r, where $1 \leqq r < j$. Hence S is not suitable, which is a contradiction to our original assumptions. Therefore (b) is true, and the proof of our theorem is now complete.

4. The theorem of the preceding section enables one to compute the value of $N(m, k)$ for any k between

$$\bar{m} \equiv \left[\frac{m}{[m^{1/2}]}\right] + [m^{1/2}] - 1$$

and m, inclusive. However, the method of proving statement (a) in the above section can be used to show that, if $k < \bar{m}$, then

$$N(m, k) \leqq m - [m^{1/2}].$$

In particular, if $m = n^2+n$ or n^2+2n, so that $\bar{m} = 2n$ or $2n+1$, respectively, then

(I) $\quad N(n^2 + n, 2n - 1) \leqq n^2, \qquad N(n^2 + 2n, 2n) \leqq n^2 + n.$

Direct computation by means of the theorem will give

(II) $\quad N(n^2 + n - 1, 2n - 1) = n^2, \quad N(n^2 + 2n - 1, 2n) = n^2 + n.$

Taking into account Lemma 1 of §2, (I) and (II) together show the correctness of the following theorem, which is a rather slight extension of the previous one.

THEOREM II. *If* $m=n^2+\alpha n$, $\alpha=1$ *or* 2, *then*

$$N(m-1, 2n+\alpha-2) = N(m, 2n+\alpha-2) = m-n.$$

5. An application. The "dimension" of a partially ordered set has been considered in a paper by Miller and the author;[3] since this term was used in a sense different from that employed by others,[4] we give its definition here.

If P and L are, respectively, a partial ordering and a (simple) linear ordering of a set S, then L will be called a linear extension of P whenever the following is true: if x and y are elements of S such that $x<y$ in P, then $x<y$ in L. For a given P (on S), let K be a collection of linear extensions of P which has the property that, if x and y are elements of S which are *not* ordered in P, then there exist in K two extensions, L' and L'', such that $x<y$ in L' and $y<x$ in L''. Such a collection K will be said to "realize" the partial order P. By the dimension of a partial order P, denoted by dim P, we shall mean the smallest cardinal number n such that P is realized by a collection of n extensions of itself.

It is known[5] that every P has a dimension, which is a finite number if the set on which P is defined is finite. The dimension of certain special partial orders has been computed. For example, the partial order defined on the family of all subsets of a set of power n, with set inclusion as the ordering relation, has dimension n.[6] Similarly, consider the family of all r-subsets and s-subsets of M, where $1 \leqq r < s < m$; let $P(m, r, s)$ denote the partial order defined on this family by using, as above, set inclusion as the ordering relation. It has been shown[7] that

(a) $$\dim P(m, 1, m-1) = m.$$

In terms of the results obtained in this paper, the following generalization of (a) may now be stated.

THEOREM III. *If* $1<k<m$, *then*

$$\dim P(m, 1, k) = N(m, k+1).$$

PROOF. For convenience, put $P(m, 1, k)=P$, $\dim P=s$, $N(m, k+1)=N$, and let

$$L \equiv \{L_1, L_2, \cdots, L_s\}$$

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[3] See [1], in bibliography at the end.
[4] For example, by G. Birkhoff in his book *Lattice theory*.
[5] See [1, p. 601].
[6] See [3, p. 509].
[7] See [1, p. 604].

be a set of s linear extensions of P which realize P. For any i between 1 and s inclusive, define A_i as the arrangement of M which results when, in L_i, we discard all the symbols which represent k-subsets, and replace all the other symbols (which represent 1-subsets) by their corresponding integers. The set

$$A \equiv \{A_1, A_2, \cdots, A_s\}$$

is thus a set of s arrangements of the elements of the set M. Now, let

$$a_1, a_2, \cdots, a_k, a_{k+1}$$

be any $(k+1)$-subset of M. The k-subset $x \equiv (a_1, a_2, \cdots, a_k)$ and the 1-subset $y \equiv (a_{k+1})$ are not ordered in P; hence, in one of the extensions of L, we must have $x < y$, say

$$x < y \quad \text{in} \quad L_i.$$

Since L_i is an extension of P, we must also have

$$(a_j) < x \quad \text{in} \quad L_i, \qquad j = 1, 2, \cdots, k,$$

and therefore

$$(a_j) < (a_{k+1}) \quad \text{in} \quad L_i, \qquad 1 \leqq j \leqq k.$$

It now follows, from the definition of A_i, that a_{k+1} follows a_j in A_i, $1 \leqq j \leqq k$. In other words, the set A is $(k+1)$-suitable, and, since the number of arrangements in A is s, we have

$$N(m, k+1) \leqq s = \dim P(m, 1, k).$$

To complete the proof of our theorem, it suffices to show that

$$\dim P(m, 1, k) \leqq N(m, k+1).$$

For this purpose, let

$$A \equiv \{A_1, A_2, \cdots, A_N\}$$

be a $(k+1)$-suitable set of N arrangements of the elements of M. Transform any one of these arrangements, say

$$A_i: \quad a_1, a_2, \cdots, a_m,$$

into a linear ordering of 1-subsets and k-subsets of M in the following manner. The symbols a_r will be re-interpreted to represent the corresponding 1-subsets of M, and these a's will be ordered according to the order of their appearance in A_i (from left to right). If x represents a k-subset of M, we insert x into A_i so that x follows immediately after the last symbol which represents an element of x. If $x, y, \cdots, z$

are all the k-subsets each of which is "entitled" to the same position, then they are inserted as a unit in that position, the order among them being arbitrary. [To illustrate, if the transformed A_i is denoted by L_i, then

$$L_i\colon\quad a_1 < a_2 < \cdots < a_k < w < a_{k+1} < x < y < \cdots < z < \cdots$$

where w is the k-subset whose elements are $a_1, a_2, \cdots, a_k$, and $x, y, \cdots, z$, the several k-subsets each of which contains a_{k+1} and some $k-1$ of the first k elements of A_i.] It is obvious that the L_i so obtained by transforming A_i is a linear extension of P, so that

$$L \equiv \{L_1, L_2, \cdots, L_N\}$$

is a set of extensions of P.

Now, let x and y be any two subsets of M which are not compared in P. There are three possibilities:

(a) x and y are both k-subsets;

(b) x and y are both 1-subsets;

(c) x is a k-subset, and y is a 1-subset which is not contained in x.

We discuss in detail only case (a), which is perhaps the most difficult. Let r and s be elements of M such that r is an element of x which does not belong to y, and s is an element of y which does not belong to x. The union of r and all the elements of y is a $(k+1)$-subset, and since A is $(k+1)$-suitable, there will exist A_i in A in which r will follow all the elements of y. From the manner of construction of L_i, it clearly follows that

$$y < x \quad \text{in} \quad L_i.$$

Similarly, there will exist L_j in L, $j \neq i$, such that $x < y$ in L_j. Cases (b) and (c) can be handled in analogous fashion, with the same final result. But all this means that L realizes P. Since the number of extensions in L is N, we have

$$\dim P(m, 1, k) \leqq N = N(m, k+1).$$

As already pointed out, this completes the proof of Theorem III.

6. The partial ordering P' (of a set S') is said to be isomorphic to P (on S) if there exists a 1-1 mapping of S onto S' which preserves the partial ordering; P' is said to be dual to P if the 1-1 mapping inverts the partial ordering, that is, if x and y are elements of S whose images in S' are x' and y', then $x' < y'$ in P' if, and only if, $y < x$ in P. It is fairly obvious that

(a) $$\dim P = \dim P'$$

whenever P' is either isomorphic with or dual to P.

Suppose now we have $1 \leqq r < s < m$. Every r-subset x of M determines uniquely an $(m-r)$-subset of M, namely, its complement $X' \subseteq M - x$. Likewise, if y is an s-subset of M, then $y' \equiv M - y$ is an $(m-s)$-subset of M, and $y' \subset x'$ if, and only if, $x \subset y$. In other words, there is a "natural" 1-1 mapping of the family of all r- and s-subsets of M onto the family of all $(m-r)$- and $(m-s)$-subsets of M, and in terms of this mapping we see that $P(m, r, s)$ is dual to $P(m, m-s, m-r)$. Because of (a), the following theorem has thus been demonstrated:

THEOREM IV. *If* $1 \leqq r < s < m$, *then*

$$\dim P(m, r, s) = \dim P(m, m - s, m - r).$$

Finally, we may state the following theorem:

THEOREM V. *If* $1 < k < m$, *then*

$$\dim P(m, m - k, m - 1) = N(m, k + 1).$$

This follows from the preceding theorem with $r = 1$, if we also take Theorem III into account.

7. **Conclusion.** The methods used in this paper afford no means of computing $N(m, k)$ when k is small in comparison with m, although crude estimates in such a case can be obtained by Lemma 1.[8]

Similarly, the application of the notions developed here to the problem of determining the dimension of $P(m, r, s)$ is limited to the very special cases $r = 1$ or $s = m - 1$.

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[8] The argument in the proof of theorem 4.22 in [1, p. 605] shows that $\dim P(m, 1, 2) \to \infty$ as $m \to \infty$. By Theorem III of this paper we therefore have $\lim_{m \to \infty} N(m, 3) = \infty$, and by Lemma 1 we conclude that, for fixed $k \geqq 3$, $\lim_{m \to \infty} N(m, k) = \infty$,

ON THE DEFINING FIELD OF A DIVISOR IN AN ALGEBRAIC VARIETY[1]

WEI-LIANG CHOW

In the system of algebraic geometry as developed by A. Weil in his recent book, *Foundations of algebraic geometry*, a variety U in the n-space is defined as the set of all equivalent couples (k, P), each consisting of a field k and a point P in the n-space such that the field $k(P)$ is a regular extension of k. Two such couples (k', P') and (k'', P'') are called equivalent if every finite specialization of P' over k' is also one of P'' over k'' and conversely. Any field k which enters into such a couple is called a field of definition of the variety U. It has been shown by Weil in his book that among all the fields of definition of a variety U there is a smallest one which is contained in all of them, which we shall call the defining field of the variety U. A d-cycle G in a variety U of dimension r is a finite set of simple subvarieties of dimension d in U, to each of which is assigned an integer called its multiplicity; a cycle is called positive if the multiplicity of each of its component varieties is positive. Let K be a field of definition of U. Then the G is said to be rational over K if it satisfies the following conditions: (1) each component variety of G is algebraic over K; (2) if a variety is a component of G, then all the conjugate varieties over K are also components of G with the same multiplicity; (3) the multiplicity of each component of G is a multiple of its order of inseparability. The question arises whether there is a smallest one among all the fields over which the cycle G is rational. If such a smallest field exists, we shall call it the defining field of the cycle G. One observes that since by definition every field over which the cycle G is rational must be a field of definition of the variety U, it follows that the defining field of G, if it exists, must contain the defining field of U.

The following simple example shows that in general a cycle does not have a defining field. Consider the variety U_0 consisting of the one point $(x^{1/p}, y^{1/p})$ in the 2-space S_2, where x and y are independent variables over a field k of characteristic p; this variety U_0 is a simple subvariety of S_2. The 0-cycle pU_0 is rational over both fields $k(x, y^{1/p})$ and $k(x^{1/p}, y)$; but it is not rational over the field $k(x, y^{1/p}) \cap k(x^{1/p}, y) = k(x, y)$.

In a recent discussion, Weil has communicated to me the conjecture

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[1] This note is essentially an extract from a letter of the author to Professor André Weil, dated February 3, 1949.

that *in case of a divisor in a variety there always exists a defining field.* I shall give here a proof of this conjecture, based on the method of the associated forms.[2] Let U be a variety of dimension r in the n-space and let k be its defining field. Let $F(Z, Z^{(1)}, \cdots, Z^{(r)})$ be the associated form of U, where the (Z), $(Z^{(1)})$, $\cdots$, $(Z^{(r)})$ are $r+1$ sets of $n+1$ indeterminates each. It can be easily shown that this form $F(Z, Z^{(1)}, \cdots, Z^{(r)})$ is rational over k and absolutely irreducible. We can restrict ourselves to positive cycles in U, since every cycle rational over a field K is the difference of two positive cycles which are rational over K. To each positive d-cycle G in U (since it is also a positive d-cycle in the n-space) there corresponds an associated form $f(Z, Z^{(1)}, \cdots, Z^{(d)})$; let (f) be the set of coefficients of this form. Then the field $k((f))$ is contained in every field over which the cycle G is rational.[3] It is clear that if the cycle G is rational over $k((f))$, then this field is its defining field. Hence, to prove our conjecture, it is necessary only to show that in case $d=r-1$ the divisor G is rational over the field $k((f))=K$.

Without loss of generality we can assume that the form $f(Z, Z^{(1)}, \cdots, Z^{(r-1)})$ is irreducible over K; for otherwise we need only to repeat the same argument for each irreducible factor of this form over K. Over the algebraic closure of K the form $f(Z, Z^{(1)}, \cdots, Z^{(r-1)})$ will then be the p^eth power of a product of distinct irreducible forms, all conjugate with respect to each other over K; this p^e is by definition the multiplicity of each component of the divisor G. Let (x) be a generic point of one of the components of G over K (it does not matter which, as they are all conjugates over K). We have to prove that the field $K(x)$ has the order of inseparability $\leqq p^e$ over K.

Let $z_j^{(i)}$, $i=1, \cdots, r$, $j=1, \cdots, n$, be nr independent variables over $K(x)$, and set $z_0^{(i)} = -\sum_{j=1}^{n} x_j z_j^{(i)}$. Since the point (x) has the dimension $r-1$, the $r(n+1)-1$ elements $z_j^{(i)}$, for all i, j except $i=r$, $j=0$, are independent variables over K. Hence, the form $f(Z, z^{(1)} \cdots, z^{(r-1)})$ is exactly the p^eth power of a product of distinct linear forms. Let $f(Z_0)$ be the polynomial obtained from $f(Z, z^{(1)}, \cdots, z^{(r-1)})$ by

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[2] See Chow and van der Waerden, Math. Ann. vol. 113 (1937) pp. 692–704. Strictly speaking, the concept of the associated form is defined only for varieties and positive cycles in a projective space. However, since the affine n-space can be extended to a projective space and every variety in the former can be extended to a variety in the latter in a unique way, it is easily seen that our application of this concept to varieties and positive cycles in the affine n-space is justified.

[3] If $(f)=(f_0, f_1, \cdots, f_t)$ is a set of quantities, not all zero, then the field $K((f))$ is the extension of K obtained by the adjunction of all the ratios of any two non-vanishing elements of the set (f).

setting $Z_j = z_j^{(r)}$, $j=1, \cdots, n$; then $f(Z_0)$ is the p^eth power of a separable polynomial. Since $z_0^{(r)}$ is a root of this polynomial $f(Z_0)$, it follows that $z_0^{(r)}$ has the order of inseparability p^e over the field $K(z^{(1)}, \cdots, z^{(r-1)}; z_1^{(r)}, \cdots, z_n^{(r)})$. Denote by K^* the field obtained from K by the adjunction of the nr element $z_j^{(i)}$, $i=1, \cdots, r$, $j=1, \cdots, n$. Then our result shows that $K(z^{(1)}, \cdots, z^{(r)})$ has the order of inseparability not greater than p^e over K^*. Consider now the form $F(Z, z^{(1)}, \cdots, z^{(r)})$; it dissolves over the algebraic closure of $K(z^{(1)}, \cdots, z^{(r)})$ into a product of distinct linear forms each of which corresponds to an intersection of U with the linear variety defined by the equations $\sum_{j=0}^{n} z_j^{(i)} X_j = 0$, $i=1, \cdots, r$. For, the point (x) is a simple point of U (by the definition of a divisor) and the linear variety is a generic one containing the point (x). Since the point (x) is one of these intersections, therefore the linear form $\sum_{j=0}^{n} x_j Z_j$ is a simple factor of the form $F(Z, z^{(1)}, \cdots, z^{(r)})$; it is, in fact, the only factor independent of the variables $z_j^{(i)}$, $i=1, \cdots, r$; $j=1, \cdots, n$. Therefore, the point (x) is rational over the field $K(z^{(1)}, \cdots, z^{(r)})$, and hence $K^*(x) = K(z^{(1)}, \cdots, z^{(r)})$. This shows that $K^*(x)$ has the order of inseparability not greater than p^e over K^*; and since $K(x)$ and K^* are linearly disjoint over K, it follows that $K(x)$ has also the order of inseparability not greater than p^e over K.

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THE SPACE H^p WITH $0<p<1$

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1. Introduction. The space H^p is defined to be the class of all functions $f\equiv f(z)$ which are regular on the interior of the unit circle and such that $\sup_{0\leqq r<1}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta<\infty$. For arbitrary $f\in H^p$ we then define $\|f\|\equiv\sup_{0\leqq r<1}((1/2\pi)\int_0^{2\pi}|f(re^{i\theta})|^p d\theta)^{1/p}$, which we shall call the "norm of f." We note here that the norm so defined is not a norm in the usual sense when $0<p<1$, in view of the fact that the triangular law does not hold in this case. However, A. E. Taylor [1][1] has shown that when $1\leqq p$, H^p is a Banach space, hence in that case it really is a norm. When $0<p<1$, H^p becomes a complete, perfectly separable, linear topological space under the topology: $U\subseteq H^p$ is open in case for arbitrary $f_0\in U$ it is true $\exists r>0\ni E_f[\|f-f_0\|<r]$ (a sphere of radius r about f_0) lies in U. In fact H^p is linearly homeomorphic to a closed subspace of $L^p[0, 2\pi]$, where M. M. Day [2] has defined the topology of $L^p[0, 2\pi]$. M. M. Day [2] has shown that $(L^p[0, 2\pi])^*$, the class of linear functionals on $L^p[0, 2\pi]$, is void save for the zero functional, whence one might suspect the same to be true of $(H^p)^*$, the class of linear functionals on H^p. This, however, is not the case. Defining $\gamma_{n,z}(f)\equiv f^{(n)}(z)/n!$, we show that $\gamma_{n,z}\in(H^p)^*$ for all $z\ni|z|<1$ and $n=0, 1, \cdots$. Hence, not only does H^p have nonzero functionals, but there exists a countable collection of such functionals which will distinguish members of H^p. As a result it then makes sense to speak of weak convergence in H^p, where weak convergence is defined in H^p just as in any normed linear space. It is shown that any family $\mathfrak{F}\subseteq H^p$ for which $\gamma(f)$ is bounded in $f\in\mathfrak{F}$ for each $\gamma\in(H^p)^*$ has the property that when considered as a family defined on any compact subset of the unit circle it is a uniformly bounded family. As a direct result of this, we then find that a weakly convergent sequence in H^p converges uniformly on any compact subset of the unit circle to its weak limit.

2. Inequalities. The following inequalities, valid for $0<p<1$, are easily established by a consideration of the function $(1+t^p)/(1+t)^p$, $t\geqq 0$:

$$(1)\qquad (a+b)^p\leqq a^p+b^p,\qquad a\geqq 0,\ b\geqq 0,$$

$$(2)\qquad (a^p+b^p)^{1/p}\leqq 2^{(1-p)/p}(a+b),\qquad a\geqq 0,\ b\geqq 0.$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

However, Theorem 19, p. 28 of [3] yields (1), while Theorem 13 (2.8.4), p. 24 of the same reference yields (2) by letting k and b of that reference be respectively p and unity, and the a's of that reference be a and b of (2) above. From the definition of the norm in H^p, $0<p<1$, and (1) it is then clear that if f and g are in H^p we have

$$\|f+g\|^p \leq \|f\|^p + \|g\|^p. \tag{3}$$

Inequality (2) yields $(\|f\|^p+\|g\|^p)^{1/p} \leq 2^{(1-p)/p}(\|f\|+\|g\|)$, whence we obtain from (3)

$$\|f+g\| \leq 2^{(1-p)/p}(\|f\| + \|g\|). \tag{4}$$

It is to be noted that (3) insures the continuity of the norm in H^p, $0<p<1$. Unless otherwise stated, we shall henceforth assume $0<p<1$.

3. The topology of H^p. In the introduction it was mentioned how we topologize H^p so that it becomes a linear topological space (a vector space which is a Hausdorff space in which addition and scalar multiplication are continuous operations in the space). We shall use here those axioms set forth by W. Sierpiński [4] in defining a Hausdorff topological space.

THEOREM 1. *H^p is a linear topological space.*

PROOF. That H^p is a vector space is implied by either of the inequalities (3) or (4) and the obvious homogeneity of the norm. It is clear that the union of any number of open sets is open and the intersection of any two open sets is open. We note that the spheres in H^p are open sets as a direct consequence of (3). As a result of this, inequality (3) or (4), and the fact that if the norm of an element of H^p vanishes then the element is necessarily the zero element of H^p, we conclude that the Hausdorff separation axiom holds in H^p. The continuity of addition and scalar multiplication in H^p is a direct consequence of (3) and (4).

THEOREM 2. *If $f \in H^p$, then*

$$|f(z)| \leq \frac{\|f\|}{(1-|z|)^{1/p}}, \qquad |z|<1.$$

PROOF. We may suppose $f \neq 0$. By virtue of F. Riesz's decomposition theorem [5] we may write $f(z)=g(z)h(z)$, where $h(z)$ is regular and bounded by unity on $|z|<1$, and $g \in H^p$ with $\|g\|=\|f\|$ and $g(z) \neq 0$ on $|z|<1$. Thus it is clear that $[g(z)]^p$ may be defined properly so that it is a member of H^1. By Cauchy's integral formula

$$[g(z)]^p = \frac{1}{2\pi}\int_0^{2\pi} \frac{[g(\rho e^{i\theta})]^p \rho e^{i\theta} d\theta}{\rho e^{i\theta} - z}, \qquad |z| < \rho < 1.$$

Thus

$$|g(z)|^p \leqq \frac{\rho}{\rho - |z|}\left(\frac{1}{2\pi}\int_0^{2\pi} |g(\rho e^{i\theta})|^p d\theta\right)$$

$$\leqq \frac{\rho}{\rho - |z|}\cdot\|g\|^p = \frac{\rho}{\rho - |z|}\cdot\|f\|^p.$$

Thus $|f(z)|^p \leqq (1/(1-|z|))\cdot\|f\|^p$, the desired conclusion.

An arbitrary linear topological space X will be said to be complete in case every Cauchy sequence in X has a limit in X ($\{x_n\}\subseteq X$ is a Cauchy sequence in case $\lim_{n,m\to\infty}(x_n - x_m) = 0$).

THEOREM 3. *H^p is complete.*

A proof of the completeness of H^p for $1 \leqq p$ is to be found in A. E. Taylor's paper [1], and the proof is trivially modified for our case when $0 < p < 1$ when we note the preceding theorem.

M. M. Day [2] has defined a topology over $L^p[0, 2\pi]$ which is equivalent to the following definition: a set $U \subseteq L^p[0, 2\pi]$ is said to be open in case for arbitrary $\phi_0 \in U$ it is true $\exists r > 0 \ni E_\phi[\|\phi - \phi_0\| < r]$ lies in U, where for arbitrary $\phi \in L^p[0, 2\pi]$, $\|\phi\| \equiv ((1/2\pi)\cdot\int_0^{2\pi}|\phi(\theta)|^p d\theta)^{1/p}$. That $L^p[0, 2\pi]$ is a linear topological space is seen just as it was that H^p is a linear topological space.

THEOREM 4. *H^p is equivalent to a closed subspace of $L^p[0, 2\pi]$ (that is, $\exists$ an algebraic isomorphism Γ of H^p onto a closed subspace of $L^p[0, 2\pi]$, the isomorphism being norm preserving).*

PROOF. F. Riesz [1] has shown that if $f \in H^p$, then $\lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere on $[0, 2\pi]$, and moreover, if we designate $\lim_{r\to 1} f(re^{i\theta})$ as $f(e^{i\theta})$ we have $f(e^{i\theta}) \in L^p[0, 2\pi]$ and $\|f\| = \|f(e^{i\theta})\|$, that is, f as a member of H^p has the same norm as $f(e^{i\theta})$ has as a member of $L^p[0, 2\pi]$. We then define $\Gamma(f) \equiv f(e^{i\theta})$. That Γ is 1:1 is clear from the norm preserving property of Γ. That the range of Γ is closed is clear since a Cauchy sequence in the range of Γ maps into a Cauchy sequence in H^p.

THEOREM 5. *H^p is perfectly separable (satisfies the second axiom of countability).*

PROOF. The proof that $L^p[0, 2\pi]$, $0 < p < 1$, is separable is practically the same as the proof when $1 \leqq p < \infty$. As is the case in any

metric space, separability in $L^p[0, 2\pi]$ implies perfect separability, whence every subset of $L^p[0, 2\pi]$ is likewise perfectly separable. Letting $R(\Gamma)$ be the range of Γ of Theorem 4, we see that $R(\Gamma)$ is perfectly separable. Since Γ is a homeomorphism of H^p onto $R(\Gamma)$, then H^p is perfectly separable.

4. **Linear functionals.** As in any normed linear space we have the fact that if γ is a distributive functional on H^p, then $\gamma \in (H^p)^*$ if, and only if, γ is bounded on the unit sphere in H^p. We then topologize $(H^p)^*$ just as is the conjugate space of any normed linear space, that is, we define, for arbitrary $\gamma \in (H^p)^*$, $\|\gamma\| = \sup_{\|f\|=1} |\gamma(f)|$, and, as in the case of an arbitrary normed linear space, we find that $(H^p)^*$ is a Banach space.

THEOREM 6. *$\gamma_{n,z} \in (H^p)^*$, $n = 0, 1, \cdots$, $|z| < 1$, and*

$$\|\gamma_{0,z}\| \leqq \frac{1}{(1-|z|)^{1/p}},$$

$$\|\gamma_{n,z}\| \leqq \frac{\rho_{n,z}}{(\rho_{n,z}-|z|)^{n+1}(1-\rho_{n,z})^{1/p}}, \qquad n > 0,$$

where

$$\rho_{n,z} \equiv [|z|(1-p) + pn + \{[|z|(1-p) + pn]^2 + 4p|z|(pn+1)\}^{1/2}]/2(pn+1).$$

PROOF. We may suppose $n > 0$ since the case where $n = 0$ is merely a direct consequence of Theorem 2. Let $|z| < \rho < 1$. Then, by Cauchy's integral formula for the derivatives of a regular function we have $\gamma_{n,z}(f) = (1/2\pi)\int_0^{2\pi} f(\rho e^{i\theta})\rho e^{i\theta} d\theta/(\rho e^{i\theta} - z)^{n+1}$, whence $|\gamma_{n,z}(f)| \leqq (\rho/(\rho - |z|)^{n+1}) \cdot (\|f\|/(1-\rho)^{1/p})$, by virtue of Theorem 2. Thus it is then clear that $\gamma_{n,z} \in (H^p)^*$ and if $|z| < \rho < 1$, then $\|\gamma_{n,z}\| \leqq \rho/(\rho - |z|)^{n+1}(1-\rho)^{1/p}$. The final conclusion of our theorem is then obtained by minimizing the function $\rho/(\rho - |z|)^{n+1}(1-\rho)^{1/p}$ on the interval $|z| < \rho < 1$.

COROLLARY 1. *There exists a countable collection of linear functionals $\{\eta_n\}$ on H^p having the property that if $f \in H^p$ is not zero, then $\exists n \ni \eta_n(f) \neq 0$.*

PROOF. We need only choose $\eta_n \equiv \gamma_{n,z}$, where z is fixed. Another collection $\{\eta_n\}$ could be $\eta_n \equiv \gamma_{0,z_n}$, where $|z_n| < 1$ and $\{z_n\}$ is a sequence of distinct points having at least one limit point in the interior of the unit circle.

THEOREM 7. *Suppose $\mathfrak{F}$ is a family of elements of $H^p \ni \gamma(f)$ is bounded on $\mathfrak{F}$ for each fixed $\gamma \in (H^p)^*$. Then $\exists M > 0 \ni$*

(a) $$|\gamma(f)| \leqq M\|\gamma\|,$$

(b) $$|f(z)| \leqq \frac{M}{(1-|z|)^{1/p}},$$

(c) $$|\gamma_{n,s}(f)| \leqq M\left\{\frac{\rho_{n,s}}{(\rho_{n,s}-|z|)^{n+1}(1-\rho)^{1/p}}\right\}, \qquad n > 0,$$

for all $f \in \mathfrak{F}$.

PROOF. In order to prove our theorem we shall have to make use of Theorem 11, p. 19 of [6], which is:

Suppose E is a metric space, $G \subseteq E$ is of second category in E, and suppose that $\Phi = \{\phi\}$ is a family of real-valued and continuous functions on E for which $\phi(x)$ is bounded above with respect to ϕ for each fixed $x \in G$. Then $\exists$ a nontrivial sphere $U \subseteq E$ on which $\{\phi(x)\}$ are bounded uniformly from above.

We shall refer to this theorem as (I). We now define $G \equiv E \equiv (H^p)^*$. Since $(H^p)^*$ is a Banach space, then $(H^p)^*$ is of second category in itself by Baire's theorem. We define $\phi_f(\gamma) \equiv |\gamma(f)|$, $f \in \mathfrak{F}$, and let $\Phi \equiv \{\phi_f\}$. That ϕ_f is continuous in γ is clear since

$$|\phi_f(\gamma) - \phi_f(\gamma')| \leqq \phi_f(\gamma - \gamma') = |\gamma(f) - \gamma'(f)| \leqq \|\gamma - \gamma'\| \cdot \|f\|.$$

We now apply (I) to obtain some $m > 0$, $r > 0$, $\tilde{\gamma} \ni$ on $E_{\gamma'}[\|\gamma' - \tilde{\gamma}\| \leqq r]$, $\phi_f(\gamma') \leqq m$. Thus we see that in particular for the functional $\gamma' \equiv (r\gamma/\|\gamma\|) + \tilde{\gamma}$, where γ is any nonzero member of $(H^p)^*$, we have $\phi_f(\gamma') \leqq m$, or merely $|\gamma'(f)| \leqq m$, or $|(r\gamma(f)/\|\gamma\|) + \tilde{\gamma}(f)| \leqq m$, or $r|\gamma(f)|/\|\gamma\| \leqq 2m$ since $\tilde{\gamma} \in E_{\gamma'}[\|\gamma' - \tilde{\gamma}\| \leqq r]$. Then $|\gamma(f)| \leqq (2m/r)\|\gamma\|$, and letting $M \equiv 2m/r$ we obtain conclusion (a). Conclusion (b) follows immediately from (a) by letting $\gamma \equiv \gamma_{0,s}$, and by noting Theorem 6. Conclusion (c) follows from (a) by letting $\gamma \equiv \gamma_{n,s}$ and by noting Theorem 6.

5. **Weak convergence.** We note that in view of Corollary 1 it makes sense to talk of weak convergence in H^p. As in any normed linear space we say that the sequence $\{f_n\} \subseteq H^p$ converges weakly to $f \in H^p$, written $f_n \to^w f$, in case $\lim_{n\to\infty} \gamma(f_n) = \gamma(f)$ for every $\gamma \in (H^p)^*$. To be sure, Corollary 1 guarantees the uniqueness of such a weak limit.

THEOREM 8. *If $f_n \to^w f$ in H^p, then $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on all compact subsets of the unit circle.*

PROOF. Since $\gamma(f_n) \to \gamma(f)$ for all $\gamma \in (H^p)^*$, then $\{\gamma(f_n)\}$ is bounded

in n for each fixed $\gamma\in(H^p)^*$. By conclusion (b) of the previous theorem $|f_n(z)|\leqq M/(1-|z|)^{1/p}$ whence $|f_n(z)|\leqq M/(1-\rho)^{1/p}$ on $|z|\leqq\rho<1$. Moreover $\gamma_{0,z}(f_n)\to\gamma_{0,z}(f)$, or what is the same thing $f_n(z)\to f(z)$ on $|z|<1$. Thus by Vitali's theorem $\lim_{n=\infty} f_n(z)=f(z)$ uniformly on all compact subsets of $|z|<\rho$, where $\rho<1$, and hence $\lim_{n=\infty} f_n(z)=f(z)$ uniformly on all compact subsets of $|z|<1$.

THEOREM 9. *Suppose* $\{f_n\}$ *is a bounded sequence of elements of* $H^p\ni\lim_{n=\infty}\gamma_{k,0}(f_n)=\gamma_{k,0}(f)$ *for all* $k=0, 1, \cdots$. *Then* $\lim_{n=\infty} f_n(z)=f(z)$ *uniformly on all compact subsets of the unit circle.*

PROOF. Let $\epsilon>0$ be given. Consider

$$|f_n(z)-f(z)| = \left|\sum_{k=0}^{\infty}\gamma_{k,0}(f_n-f)z^k\right| \leqq \left|\sum_{k=0}^{N}\gamma_{k,0}(f_n-f)z^k\right| + \left|\sum_{k=N+1}^{\infty}\gamma_{k,0}(f_n-f)z^k\right|, \qquad |z|\leqq\rho<1.$$

Noting Theorem 6 we find that $\rho_{k,0}=pk/(pk+1)$, whence $\|\gamma_{k,0}\|\leqq((pk+1)/pk)^k(pk+1)^{1/p}$. It is easily verified that $\sum_{k=1}^{\infty}((pk+1)/pk)^k\cdot(pk+1)^{1/p}\rho^k$ converges. We now choose $N\ni M\sum_{k=N+1}^{\infty}((pk+1)/pk)^k\cdot(pk+1)^{1/p}\rho^k<\epsilon/2$, where M is a bound on $\{\|f_n-f\|\}$. Thus, since $|\gamma_{k,0}(f_n-f)|\leqq\|\gamma_{k,0}\|\cdot\|f_n-f\|\leqq M((pk+1)/pk)^k(pk+1)^{1/p}$, we find that $\left|\sum_{k=N+1}^{\infty}\gamma_{k,0}(f_n-f)z^k\right|<\epsilon/2$. We now choose $\overline{N}\ni n>\overline{N}$ implies $\left|\sum_{k=0}^{N}\gamma_{k,0}(f_n-f)z^k\right|<\epsilon/2$ for all $z\ni|z|\leqq\rho$, as can well be accomplished since $\gamma_{k,0}(f_n-f)\to0$ by hypothesis. Thus, indeed $|f_n(z)-f(z)|\leqq\epsilon/2+\epsilon/2=\epsilon$ on $|z|\leqq\rho$ for all $n>\overline{N}$, which concludes the proof of the theorem.

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COVERING BY DISMEMBERED CONVEX DISCS

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Let us consider a great number of convex discs not "too different" from one another about which, apart from their total area, no further data are known.

What can be said about the area of the largest square (or any other domain of given shape) which can be covered with the aid of these discs if we are allowed to cut each of them into a given number of suitable pieces?

Analogously, we can raise the problem of estimating the area of the smallest square into which the pieces of the discs can be placed without overlapping if each disc is allowed to be cut into a given number of convenient pieces.

These are, roughly expressed, the problems we intend to deal with in the present note.

1. **Notations and definitions.** For domains and for their area the same notation will be used.

Let S be an infinite set of convex discs $d_1, d_2, \cdots$ the set of the radii of the incircles and circumcircles of which has a positive lower and a finite upper bound r and R, respectively. We shall call such a set a *normal set.*

Let $s_n = d_1 + \cdots + d_n$ be the total area of the discs $d_1, \cdots, d_n$, t_n ($\leqq s_n$) the area of the largest domain similar to a given convex domain t which can be completely covered by $d_1, \cdots, d_n$, and T_n ($\geqq s_n$) the area of the smallest domain similar to t into which the discs $d_1, \cdots, d_n$ can be placed without overlapping.

Consider the positive values less than or equal to 1,

$$e = e(S) = \liminf t_n/s_n,$$

$$E = E(S) = \liminf s_n/T_n,$$

which can easily be shown to be independent of the choice of the domain t. We shall call *e the economy of the set S with regard to covering, E the economy of the set with regard to packing.*

Let S be a set of congruent triangles, quadrangles, or hexagons having a center of symmetry, or any other set of congruent polygons by which the plane can be tesselated. For such a set S we have obviously $e(S) = E(S) = 1$.

A nontrivial example is furnished by a set of equal circles, in which

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case—according to researches of R. Kershner [1][1] and A. Thue [1,2], respectively—$e=27^{1/2}/2\pi=0.8269\ldots$ and $E=\pi/12^{1/2}=0.9069\ldots$ holds. Various generalizations or new proofs of these equalities can be found, for example, in the papers of H. Hadwiger [1], B. Segre-K. Mahler [1], and L. Fejes Tóth [1, 2, 3].

Now we derive from S a new set by dividing each disc d_i into k convex pieces $d_i^1, \cdots, d_i^k$. The new set $d_1^1, \cdots, d_1^k, d_2^1, \cdots, d_2^k, \cdots$ ceases generally to be a normal set but it is easy to show that it has economies e_k ($\geqq e$) and E_k ($\geqq E$) in the above sense independent of the domain t.

In what follows we shall choose t to be a square.

2. **Results.** Let us now announce our results, which read as follows. *The members of an arbitrary normal set of convex discs can always be cut into $k(=1, 2, \cdots)$ convex pieces so that the derived set of pieces has economy with regard to covering*

$$e_k \geqq \frac{k+1}{\pi} \sin\frac{\pi}{k+1} \tag{1}$$

or so that the set of pieces has an economy with regard to packing

$$E_k \geqq \frac{\pi}{2k} \cot\frac{\pi}{2k}. \tag{2}$$

In case each disc has a center of symmetry and k is odd, the above inequalities can be improved as follows:

$$e_k \geqq \frac{k+2}{\pi} \sin\frac{\pi}{k+2}, \tag{3}$$

$$E_k \geqq \frac{\pi}{2k+2} \cot\frac{\pi}{2k+2}. \tag{4}$$

For instance, by cutting each disc into 40 suitable pieces there always can be obtained a covering by which only one-thousandth of the area of the discs is lost ($e=0.999$). Analogously, in order to obtain a packing by which only one-thousandth of the disposable area may remain unused ($E=0.999$), it will always suffice to cut each disc into 29 suitable pieces.

It seems probable that dismemberments into "essentially" smaller number of pieces than 40 and 29, respectively, will generally not suffice in order to obtain economies greater than or equal to 0.999.

[1] Numbers in brackets refer to the references cited at the end of the paper.

More precisely, it is to be expected that (1) and (2) give exact asymptotic estimates of $1-e_k$ and $1-E_k$ for large values of k.

3. **The case $k=1$.** The case $k=1$, that is, the consideration of the original set, deserves special attention. In §4, we shall show that the lower bound of the economies $e(S)$ and $E(S)$ of all normal sets S equals the lower bound of $e(\overline{S})$ and $E(\overline{S})$ of all sets $\overline{S}$ consisting of congruent convex discs. Therefore the various known inequalities concerning the densities of lattice-like arrangements of congruent discs give lower bounds of e and E concerning any normal set. Let us compare the bounds obtained in this way with our results.

The inequality (1) gives only $e_1 \geqq 2/\pi = 0.6366\ldots$, whereas (2) becomes the trivial inequality $E_1 \geqq 0$. On the other hand, results of I. Fáry [1] show that $e_1,\ E_1 > 2/3$. The exact lower bounds are unknown and their determination seems to involve considerable difficulties.

Let us now consider the case of discs having a center of symmetry.

Inequality (4) yields $E_1 \geqq \pi/4 = 0.7853\ldots$, which can be improved —by means of researches of K. Mahler [1] and the author [4]—up to $E_1 > 3^{1/2}/2 = 0.8660\ldots$. The exact bound is unknown but researches of K. Reinhardt [1] and K. Mahler [2] make it probable that the best possible estimation is $E_1 \geqq (9-32^{1/2}-\log 2)/(8^{1/2}-1) = 0.9024\cdots$. It is worth noting that the constant on the right is less than the value of E for a set of equal circular discs.

On the other hand, (3) gives the exact bound $e_1 \geqq 27^{1/2}/2\pi = 0.8269\cdots$, where the right side equals the value of e for a set of equal circles. This inequality, which expresses a remarkable extremum property of the ellipse, was previously found by the author [5].

4. **Sets of congruent discs.** Let $\bar{e}$ be the lower bound of the economies $e(\overline{S})$, where $\overline{S}$ ranges over all sets of congruent convex discs. We show that for any normal set S of convex discs $d_1, d_2, \cdots$ we have $e(S) \geqq \bar{e}$. The same fact holds concerning E.

By means of this result we have, for instance, the nontrivial fact that, for any normal set of quadrangles, $e = E = 1$.

The proof rests on a well known theorem of M. Fréchet [1] concerning compact sets, from which it follows that the members of a set of convex domains lying in a given circle can always be divided into a finite number of classes, the members of the same class having a prescribed small "deviation" from one another. The fact that the considered set of convex domains is compact follows from Blaschke's [1] well known Auswahlsatz.

It follows immediately that for any $\alpha<1$ there can be given a finite number N of convex discs $D_1, \cdots, D_N$ and a distribution of the discs $d_1, d_2, \cdots$ into N classes as well, so that if d_l is a disc belonging, say, to the ith class, then $D_i \geqq \alpha d_l$ and D_i can be completely covered by d_l.

Let us now fix the value α. Since a finite number of discs has no influence on the value $e(S)$, we can suppose that each class contains an infinite number of discs.

Let n_i be the number of the discs of the ith class having an index less than or equal to n. We shall denote the system of these discs, together with the total area of them, by s_n^i. Let furthermore t_n^i be the largest square which can be covered by s_n^i, and $\bar{t}_n^i$ the largest square which can be covered by n_i discs congruent to D_i. Since $t_n^i \geqq \bar{t}_n^i$, we have by definition of $\bar{e}$

$$\liminf t_n^i/n_i D_i \geqq \liminf \bar{t}_n^i/n_i D_i \geqq \bar{e}, \qquad i=1, \cdots, N.$$

Consequently there can be found an index ν so that for all $n>\nu$

$$t_n^i \geqq \alpha n_i D_i \bar{e} \geqq \alpha^2 \bar{e} s_n^i, \qquad i = 1, \cdots, N,$$

holds.

Let us now divide the squares t_n^1 and t_n^2 by three straight line cuts into five pieces which can be put together to make a single square. This can be effected in various ways (see, for example, Rouse Ball-Coxeter [1]). Then unite the arising square and t_n^3 in the same manner as above to a new square. By continuing this process we obtain a dissection of the squares $t_n^1, \cdots, t_n^N$ by $3(N-1)$ cuts into a certain number of parts which can be put together to form one square t_n'. The system L of the straight lines lying in t_n' which corresponds to the cuts has a total length less than $3(N-1)(2t_n')^{1/2} < 3(N-1)(2\pi R^2 n)^{1/2} = O(n^{1/2})$, where R is the upper bound of the radii of the circumcircles of the discs.

Since by the cuts a certain number of the covering discs are injured, we do not know whether the square t_n' can or cannot be covered by the discs $d_1, \cdots, d_n$. But the injured discs all lie in the parallel domain of the system L at distance $2R$, which can be covered by a number $\mu_n = O(n^{1/2})$ of circles of radius r, where r is the lower bound of the incircle radii of the discs. Consequently t_n' can be covered by the discs $d_1, \cdots, d_n, d_{n+1}, \cdots, d_{n+\mu_n}$, where $\mu_n = O(n^{1/2})$.

Denoting by t_n the largest square which can be covered by the system s_n of the discs $d_1, \cdots, d_n$, we have obviously $t_n = O(n)$. Thus

$$\liminf t_n/s_n = \liminf t_{n+p_n}/s_n \geqq \liminf t'_n/s_n \geqq \alpha^2\bar{e}.$$

Since this holds for any $\alpha<1$, we have $\liminf t_n/s_n \geqq \bar{e}$, q.e.d.

Our statement concerning the economy E can be proved analogously.

Analogous arguments show that the lower bound of the economies of the sets of pieces which arise by the most suitable dismemberments into k pieces of the discs of different normal sets equals the corresponding lower bound of the economies of the sets of the pieces generated only from sets of congruent convex discs.

Therefore we restrict ourselves in what follows to sets of congruent discs.

5. **An extremum property of the ellipse.** We shall make use of the following known results.

If u_n denotes the n-gon of the largest area inscribed in a convex domain d, U_n the n-gon of the smallest area circumscribed about d, then

$$u_n \geqq d\,\frac{n}{2\pi}\sin\frac{2\pi}{n}, \qquad n \geqq 3, \tag{5}$$

$$U_n < d\,\frac{n-2}{\pi}\tan\frac{\pi}{n-2}, \qquad n \geqq 5. \tag{6}$$

Equality holds in (5) only if d is an ellipse. On the other hand, the inequality $U_n \leqq d(n/\pi)\tan(\pi/n)$—which would express an extremum property of the ellipse analogous to (5)—does not hold.

The case $n=3$ of (5) was first proved by W. Blaschke [2]. His proof holds also for an arbitrary n, but this fact was not pointed out by him. On the other hand, Blaschke's proof of the unicity of the solution of the extremum problem in question cannot be extended immediately from $n=3$ to an arbitrary n.

Later on a more elementary new proof of (5) was given by E. Sas [1], which was completed by the author with a proof of the unicity valid for any number of vertices n.

The inequality (6) can be proved by means of a convenient modification of the proof of Sas as was shown by the author [6].

(Let us still note on this occasion that Blaschke's proof involves also the following interesting fact: In any convex body of volume v there always can be inscribed an n-verticed polyhedron the volume of which is greater than or equal to the volume of the n-verticed polyhedron of maximal volume inscribed into an ellipsoid of volume v.)

6. Proofs. As we have mentioned we can restrict ourselves to a set of discs congruent to a domain d.

Consider the polygon $p \equiv A_1A_2 \cdots A_{2k+2}$ having the largest area among the polygons of $2k+2$ vertices inscribed in d. Let us divide p—for example, by the cuts A_1A_4, $A_{2k+2}A_3$, $\cdots$, $A_{k+2}A_{k+5}$—into k convex quadrangles. We assert that the decomposition of each disc involved by the above dismemberment of p will answer the purpose.

For, denoting the largest square which can be covered by the nk quadrangles derived by the above dissection of n polygons congruent to p by τ_n, we have $\liminf \tau_n/np = 1$. But p satisfying the inequality $p/d \geqq ((k+1)/\pi) \sin(\pi/(k+1))$, we have

$$\liminf t_n/nd \geqq \liminf \tau_n/nd \geqq \frac{k+1}{\pi} \sin\frac{\pi}{k+1}.$$

This completes the proof of (1).

The inequality (3) follows from the fact that the construction of Sas yields, in the case of an even n and a domain d which has a center of symmetry, a polygon u_n satisfying (5) and having likewise a center of symmetry. Consequently we can inscribe in d a polygon p of $2k+4$ vertices having a center of symmetry so that $p/d \geqq ((k+2)/\pi) \sin(\pi/(k+2))$. Now we can decompose p into $k-1$ quadrangles and a hexagon having a center of symmetry. But since the plane can be filled out also by such hexagons, the proof runs in the same manner as above.

The proofs of (2) and (4) run analogously, by taking into consideration instead of the inscribed polygons the circumscribed ones and using instead of (5) the inequality (6).

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NECESSARY AND SUFFICIENT CONDITIONS FOR POISSON'S DISTRIBUTION

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1. **Introduction.** Poisson's law of small numbers is conventionally described as the distribution of the number of successes in a very large number of independent trials of very small individual probability of success.

The usual method of deriving Poisson's distribution is to set the individual probability of success equal to m/n in the binomial distribution of n repeated trials, and then pass to the limit as $n \to \infty$, m = constant $\geqq 0$: The probability $P_n(s)$ of exactly s successes approaches the Poisson value $P(s)$,

$$P(s) = e^{-m} m^s/s! \qquad (m \geqq 0; s = 0, 1, 2, \cdots). \tag{1.1}$$

But this method requires that at the nth stage of approximation (that is, when n trials are envisaged) the individual probabilities of success shall all be equal. This is a severe restriction, both from the theoretical point of view and from that of practical application.

In the most general case of an infinite sequence of sets of n independent trials ($n = 1, 2, 3, \cdots$), we have an infinite triangular array of probabilities $p_{n,k}$ ($k = 1, \cdots, n$; $n = 1, 2, \cdots$), where $p_{n,k}$ is the probability that the kth trial in the nth set shall succeed. If $P_n(s)$ is the probability of just s success in the nth set of trials ($0 \leqq s \leqq n$), the question is naturally raised as to when, for each s, $P_n(s) \to P(s)$ as $n \to \infty$.

The first task of this paper is to establish simple necessary and sufficient conditions for this limit.[1] The second task is to do the same when the probabilities $p_{n,k}$ keep in a pre-assigned ratio independent of n:

$$p_{n,1} : p_{n,2} : \cdots = a_1 : a_2 : \cdots, \tag{1.2}$$

where the sequence $\{a_i\}$ is given. The connection of this case with divergent non-negative series satisfying a certain anti-gap condition (that is, anti-Hadamard-gap) is made apparent. The third part is devoted to the application of our methods to concrete cases. We exhibit, incidentally, examples in which the Poisson distribution is ob-

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[1] Cf. also J. H. Curtiss, *Convergent sequences of probability distributions*, Amer. Math. Monthly (1943) pp. 97–98.

tained with such unequal $p_{n,k}$ that

$$\max_{1\leq k\leq n} p_{n,k}/\min_{1\leq k\leq n} p_{n,k} \tag{1.3}$$

becomes infinite as any given constant power of n. Information is supplied as to the order of approach to the Poisson distribution.

The following formulas will be needed; they are all evident consequences of simple probability reasoning.

If $m(n)$ is the expected number of successes in the set of n trials, we have

$$m(n) = \sum_{k=1}^{n} p_{n,k}, \tag{1.4}$$

$$m(n) = \sum_{s=0}^{n} sP_n(s). \tag{1.5}$$

The generating function of this distribution is given by

$$\phi_n(t) = \sum_{s=0}^{n} P_n(s)t^s = \prod_{k=1}^{n} [1 + (t-1)p_{n,k}]; \tag{1.6}$$

and $P_n(0)$ is easily seen to be

$$P_n(0) = \prod_{k=1}^{n} (1 - p_{n,k}). \tag{1.7}$$

For the Poisson distribution defined by (1.1), the "mean" or expected value of s is given by

$$m = \sum_{s=1}^{\infty} sP(s) = \sum_{s=0}^{\infty} \frac{s}{s!} e^{-m}m^s. \tag{1.8}$$

The generating function is

$$\phi(t) = \sum_{s=0}^{\infty} P(s)t^s = e^{m(t-1)}. \tag{1.9}$$

2. **Necessary conditions.** We now prove the following:

THEOREM I. *A necessary condition for*

$$\lim_{n\to\infty} P_n(s) = P(s) \tag{2.1}$$

is that both of the following equations hold:

$$\lim_{n\to\infty} m(n) = m, \tag{2.2}$$

$$\lim_{n\to\infty} \max_{1\leq k\leq n} p_{n,k} = 0. \tag{2.3}$$

First, we will show that

$$\liminf_{n\to\infty} m(n) \geqq m. \tag{2.4}$$

Let $\epsilon>0$ be arbitrarily given. Since series (1.8) is convergent, there exists an N such that

$$m < \sum_{s=0}^{N} sP(s) + \epsilon.$$

On the other hand, by (2.1),

$$\sum_{s=0}^{N} sP(s) = \lim_{n\to\infty} \sum_{s=0}^{N} sP_n(s).$$

But for all $n \geqq N$, (1.5) shows that

$$\sum_{s=0}^{N} sP_n(s) \leqq m(n);$$

so that, in view of the preceding inequalities,

$$\liminf_{t\to\infty} m(n) > m - \epsilon;$$

and the truth of this for arbitrary ϵ leads to (2.4).

For sufficiently large n, $p_{n,k}\neq 1$; otherwise there will be infinitely many values of n for which $P_n(0)=0$ (cf. (1.7)), which would lead, by (2.1), to $P(0)=0$, contrary to (1.1). We may therefore take logarithms in (1.7) and apply the remainder theorem, obtaining

$$\log P_n(0) = -\sum_{k=1}^{n} p_{n,k} - \sum_{k=1}^{n} \frac{p_{n,k}}{2(1-\theta_{n,k}p_{n,k})^2} \qquad (0<\theta_{n,k}<1).$$

In view of (1.4) and the fact that $(1-\theta_{n,k}p_{n,k})^2<1$, this leads to

$$\log P_n(0) \leqq -m(n) - \frac{1}{2}\sum_{k=1}^{n} p_{n,k}^2.$$

Subtracting this from $\log P(0)$ obtained from (1.1), we have

$$\begin{aligned} \log P(0) - \log P_n(0) &\geqq [m(n)-m] + \frac{1}{2}\sum_{k=1}^{n} p_{n,k}^2 \\ &\geqq [m(n)-m] + \frac{1}{2}\max_{1\leq k\leq n} p_{n,k}^2 \geqq m(n)-m. \end{aligned} \tag{2.5}$$

The third inequality contained in (2.5), taken in connection with (2.1) and (2.4), establishes (2.2). This result applied to the second inequality in (2.5), and (2.1), leads to (2.3). This proves the theorem.

3. **Sufficiency of the conditions.** We now show that the necessary conditions for the Poisson distribution of Theorem I are also sufficient. The first step is to prove the following:

THEOREM II. *If equations* (2.2) *and* (2.3) *are satisfied, then* $\phi_n(t) \to \phi(t)$ *uniformly in any given circle* $|t-1| \leqq r$ *of the plane of the complex variable* t. *Moreover, for that circle,*

$$\phi_n(t) = \phi(t) + O[m(n) - m] + O\Big(\max_{1 \leqq k \leqq n} p_{n,k}\Big). \tag{3.1}$$

Here the order symbol O denotes *uniform* order: If $\{c_n\}$ is an infinite sequence of constants and $\{\psi_n\}$ a sequence of functions defined on a common domain D, $\psi_n = O(c_n)$ signifies the existence of a constant $K(=K_D)$ such that, for all sufficiently large n, $|\psi_n| \leqq K|c_n|$ holds throughout D. In (3.1), $D:(|t-1| \leqq r)$.

In view of (2.3), for all sufficiently large n

$$|(t-1)p_{n,k}| \leqq r \max_{1 \leqq k \leqq n} p_{n,k} \leqq \frac{1}{2}. \tag{3.2}$$

This restriction upon n shall be maintained throughout the proof.

Let $\log(1+u)$ denote, for all $|u| \leqq 1/2$, that determination of the logarithm whose angle is between $\pm\pi$ (actually, it will be between $\pm\pi/6$). An elementary integration shows that

$$\log(1+u) = u - u^2 \int_0^1 \frac{(1-v)dv}{(1+uv)^2}, \tag{3.3}$$

where the path of integration is along the real axis from 0 to 1.

Taking logarithms in (1.6) and applying (3.3) (with $u = (t-1)p_{n,k}$) and (1.4), we obtain

$$\log \phi_n(t) = (t-1)m(n) - w_n(t), \tag{3.4}$$

where

$$w_n(t) = (t-1)^2 \sum_{k=1}^{n} p_{n,k}^2 \int_0^1 \frac{(1-v)dv}{[1+(t-1)p_{n,k}v]^2}.$$

Using (3.2) we find

$$|w_n(t)| \leqq |t-1|^2 \sum_{k=1}^{n} p_{n,k}^2 \int_0^1 \frac{(1-v)dv}{[1-|(t-1)p_{n,k}|v]^2}$$

$$\leqq r^2 \sum_{k=1}^{n} p_{n,k}^2 \int_0^1 \frac{(1-v)dv}{(1-1/2)^2}$$

$$= 2r^2 \sum_{k=1}^{n} p_{n,k}^2;$$

whence, finally,

$$|w_n(t)| \leqq 2r^2 \left(\max_{1 \leqq k \leqq n} p_{n,k} \right) \sum_{k=1}^{n} p_{n,k}$$

$$= 2r^2 m(n) \max_{1 \leqq k \leqq n} p_{n,k}. \tag{3.5}$$

On writing (3.4) in the form

$$\log \phi_n(t) = (t-1)m + (t-1)[m(n) - m] - w_n(t),$$

and noting that in virtue of (2.2), (2.3), (3.5),

$$(t-1)[m(n) - n] = O[m(n) - m],\ w_n(t) = O\left(\max_{1 \leqq k \leqq n} p_{n,k} \right),$$

equation (3.1), and therewith our theorem, become evident.

THEOREM III. *Equations* (2.2) *and* (2.3) *are sufficient to ensure that* (2.1) *hold uniformly in* s; *and, moreover, that*

$$P_n(s) = P(s) + O[m(n) - m] + O\left(\max_{1 \leqq k \leqq n} p_{n,k} \right). \tag{3.6}$$

Here O denotes uniform order for s ranging over the domain D: $(0 \leqq s \leqq \infty)$.

We apply Theorem II with any fixed $r \geqq 2$ and denote by C the circle $|t-1| = r$. Expressing $P_n(s)$ and $P(s)$ as contour integrals about C by the Cauchy integral formula, and applying (3.1), we obtain the conclusion (3.6) immediately.

Thus equations (2.2) and (2.3) are necessary and sufficient for the Poisson distribution, (2.1); they shall be called the *Poisson conditions*.

We remark that when $m > 0$, a slight modification in the approximating process leads to a simpler asymptotic relation than (3.6): If $n_{n,k}$ satisfy the Poisson conditions with $m > 0$, so will

$$p'_{n,k} = \frac{m}{m(n)}\, p_{n,k}, \tag{3.7}$$

once n is sufficiently large. And since the corresponding $m'(n) = m$, we obtain the same Poisson distribution, only with (3.6) replaced by

$$P'_n(s) = P(s) + O(\max_{1 \leq k \leq n} p'_{n,k}). \tag{3.8}$$

4. **The case of proportional probabilities.** Many practical applications suggest the consideration of the array $\{p_{n,k}\}$ which satisfies, in addition to the Poisson conditions, the further *proportionality condition* (1.2). Algebraically, this means that for all i, j from 1 to n, and for all n,

$$a_i p_{n,j} = a_j p_{n,i}. \tag{4.1}$$

Evidently we can assume that the a's are non-negative and not all zero.

There is no essential loss in generality in assuming $a_1 \neq 0$. For there is always a first a, call it a_{N+1}, which is not zero. Applying (4.1) with $i = N+1$ and $j \leqq N$, we see that for all $n > N$,

$$p_{n,1} = p_{n,2} = \cdots = p_{n,N} = 0.$$

Discard the first N sets of trials. In each of the infinitude of remaining sets, ignore the first N trials: All these have probability zero. In other words, consider the infinite triangular array $\{p'_{n,k}\}$, where

$$p'_{n,k} = p_{N+n,N+k} \qquad (1 \leqq k \leqq n;\ n = 1, 2, \cdots).$$

This is the sub-array of the original array $\{p_{n,k}\}$, lying to the right of the vertical line drawn through the column of quantities $p_{n,N}$ $(n = N, N+1, \cdots)$. On setting $a'_k = a_{N+k}$, the proportionality conditions (4.1) (p and a being accented) are still valid; but now $a'_1 \neq 0$. Furthermore, if $\{p_{n,k}\}$ satisfies the Poisson conditions, so will $\{p'_{n,k}\}$, and *with the same mean* m.

THEOREM IV. *Let* $\{p_{n,k}\}$ *satisfy the Poisson conditions with* $m > 0$, *and also the proportionality conditions* (4.1) *with* $a_1 > 0$. *Then the series* $a_1 + a_2 + \cdots$ *diverges and satisfies the "anti-gap condition"*

$$\lim_{n \to \infty} \frac{a_n}{s_n} = 0, \qquad s_n = a_1 + \cdots + a_n. \tag{4.2}$$

Furthermore, for all sufficiently large n,

$$p_{n,k} = m(n) a_k / s_n \qquad (1 \leqq k \leqq n). \tag{4.3}$$

Suppose that, for a particular n, $p_{n,1}=0$. Applying (4.1) with $i=1$ and $j>1$, we see that $p_{n,j}=0$; so that $m(n)=0$. This cannot occur for infinitely many values of n, since $m\neq 0$. Therefore, for all sufficiently large n, $m(n)\neq 0$ and $p_{n,1}\neq 0$. We now confine our attention to such n.

Set $i=1$, $j=k$, in (4.1) and solve for a_k:

$$a_k = a_1 p_{n,k}/p_{n,1} \qquad (k = 1, 2, \cdots, n); \tag{4.4}$$

whence

$$s_n = a_1 m(n)/p_{n,1}. \tag{4.5}$$

It follows from the Poisson conditions ($m>0$) that $s_n\to\infty$. Further, (4.4) and (4.5) show that

$$\frac{a_k}{s_n} = \frac{p_{n,k}}{m(n)} \qquad (1 \leqq k \leqq n), \tag{4.6}$$

from which the Poisson conditions lead to (4.2). And (4.3) follows from (4.6). This completes the proof.

THEOREM V. *Let $a_1+a_2+\cdots$ be any divergent non-negative term series satisfying the anti-gap condition* (4.2); *and let $m(n)$ be any non-negative sequence converging to any $m>0$. Then if $p_{n,k}$ are defined by* (4.3), *they will satisfy the Poisson and proportionality conditions* (4.1).

Equations (4.1) and (2.2) being evident, we have but to establish (2.3). Let $\mu(n)$ be the subscript i of that a_i which is the (first) maximum in the set $\{a_1, \cdots, a_n\}$. Clearly

$$\max_{1\leq k\leq n} p_{n,k} = \frac{m(n)a_{\mu(n)}}{s_n}.$$

Now $\mu(n)$, which is nondecreasing as n increases, either remains bounded as $n\to\infty$ (in which case (2.3) is an evident consequence of $s_n\to\infty$) or else becomes infinite. In the latter case we have

$$\max_{1\leq k\leq n} p_{n,k} = \frac{m(n)a_{\mu(n)}}{s_{\mu(n)}}\cdot\frac{s_{\mu(n)}}{s_n} \leqq \frac{m(n)a_{\mu(n)}}{s_{\mu(n)}};$$

(2.3) is now a consequence of the anti-gap condition.

An example, of practical significance, of the Poisson and proportionality conditions is furnished by the sequence $\{c_i\}$ ($c_i\geqq 0$) having a limiting arithmetic mean m of its first n terms, as $n\to\infty$. If $p_{n,k}=c_k/n$, the Poisson and proportionality conditions ((4.1) with $a_k=c_k$) may be seen to apply, leading to (2.1). But we may generalize this type of example by extending the notion of arithmetic mean.

Let us call any sequence $\{\omega_i\}$ an *averaging sequence* if it has the three properties:

$$0 < \omega_1 \leqq \omega_2 \leqq \cdots \tag{4.7}$$

$$\lim_{n\to\infty} \omega_n = \infty, \tag{4.8}$$

$$\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = 1; \tag{4.9}$$

and call a sequence $\{c_i\}$ $(c_i \geqq 0)$ *ω-averagable* if

$$\lim_{n\to\infty} (c_1 + \cdots + c_n)/\omega_n = m. \tag{4.10}$$

Under these conditions, and if $m > 0$, it will follow that the array

$$p_{n,k} = c_k/\omega_n \tag{4.11}$$

satisfies the hypothesis of Theorem V.

Equations (2.2) and (4.1) being obvious, we have but to verify (2.3). First note that, if $s_n = c_1 + \cdots + c_n$,

$$\frac{c_n}{\omega_n} = \left(\frac{s_n}{\omega_n} - \frac{s_{n-1}}{\omega_{n-1}}\right) + \left(1 - \frac{\omega_{n-1}}{\omega_n}\right)\frac{s_{n-1}}{\omega_{n-1}}.$$

Then (4.9) and (4.10) show that $c_n/\omega_n \to 0$. Let $\mu(n)$ be the subscript i of the (first) maximum c_i in $\{c_1, \cdots, c_n\}$; clearly

$$\max_{1\leqq k\leqq n} p_{k,n} = \frac{c_{\mu(n)}}{\omega_n} = \frac{c_{\mu(n)}}{\omega_{\mu(n)}}\frac{\omega_{\mu(n)}}{\omega_n}.$$

From this it follows, using (4.8) if $\mu(n)$ is bounded, and (4.7) and $c_n/\omega_n \to 0$ when $\mu(n) \to \infty$, that (2.3) is true in all cases.

Examples of this theorem occur when $\omega_n = n^r$ $(r > 0)$, $\omega_n = \sum_{h=1}^{n} h^{-r}$ $(0 < r \leqq 1)$, $\omega_n = \log n$, and so on. But if $\omega_n = r^n$ $(r \neq 0)$, the hypothesis fails.

5. **Illustrations.** We conclude this note with three examples of the present method, the first showing its quantitative flexibility, the second and third being of practical significance; the third will also illustrate a generalization of the proportionality conditions.

EXAMPLE 1. Let $\{b_n\}$ be a sequence bounded as follows:

$$0 < \alpha \leqq b_n \leqq \beta < \infty \qquad (n = 1, 2, \cdots). \tag{5.1}$$

On setting

$$a_n = b_n/n^r \qquad (0 \leqq r \leqq 1) \tag{5.2}$$

and taking $p_{n,k}=ma_k/s_n$ ($s_n=a_1+\cdots+a_n$), we easily see that the hypothesis of Theorem V is valid. This leads to (2.1) and (4.1); and the ratio of (1.3) is between $(\alpha/\beta)n^r$ and $(\beta/\alpha)n^r$. Referring to (3.8), the use of obvious inequalities shows that

$$P_n(s) = P(s) + O\left(\frac{1}{n^{1-r}}\right) \qquad (0 \leqq r < 1);$$

$$= P(s) + O\left(\frac{1}{\log n}\right) \qquad (r = 1).$$

Similarly, if we replace (5.2) by

$$a_n = b_n n^r \qquad (r \geqq 0), \tag{5.3}$$

$p_{n,k}=ma_k/s_n$ as before. The ratio (1.3) is again between $(\alpha/\beta)n^r$ and $(\beta/\alpha)n^r$, while (3.8) leads to

$$P_n(s) = P(s) + O\left(\frac{1}{n}\right).$$

EXAMPLE 2. Let an object, specified by a coordinate x, move with x going from large negative to large positive values, and, at each crossing of an integral value, be exposed to a chance of an event which we shall call a "hit." The probabilities of various numbers s of hits is required. It is assumed that the probabilities of a hit at the various integral points are proportional to given numbers, and that the total expected number of hits has a given value m. One is interested in the case where all the risks of hit are very small but the number of danger points crossed is very large: mathematically, in a limiting process.

In practice, the situation can almost always be put in the following form: A function $f(x)$ is given for all x; it is continuous and non-negative everywhere, and approaches zero as $x\to\pm\infty$. It is bounded and has a finite number of relative maxima on $(-\infty, \infty)$. Finally, its integrals over $(0, \infty)$ and $(0, -\infty)$ diverge. The probabilities of hits at the integral points $x=k$ are proportional to $f(k)$.

Let l be any positive integer, set $n=2l+1$, and write

$$k = -l,\ -l+1, \cdots,\ -1, 0, 1, \cdots, l-1, l.$$

If, then, we take for the probability of a hit at $x=k$

$$p_{n,k} = m\frac{f(k)}{\sum_{i=-l}^{l} f(i)},$$

the conditions of our problem will be satisfied at the nth approximation. What occurs when $n \to \infty$ (that is, $l \to \infty$)?

Since $f(x)$ has but a finite number of relative maxima, a comparison of areas shows that as $l \to \infty$,

$$\sum_{i=-l}^{l} f(i) \to \infty .$$

On the other hand, since $f(x) \leqq M$ (boundedness),

$$\max_{-l \leqq k \leqq l} p_{n,k} \leqq \frac{M}{\sum_{i=-l}^{l} f(i)} \to 0$$

as $l \to \infty$. Thus the Poisson conditions are satisfied—with this immaterial change of notation for k, and the equally immaterial restriction of n to being odd. The Poisson distribution is obtained, the limiting probability of just s hits as the object moves from $-\infty$ to $+\infty$ being $P(s)$.

EXAMPLE 3. Another exposure problem is the following: The region of exposure is the fixed interval $a \leqq x \leqq b$, which the object crosses once, from left to right. On this interval a continuous non-negative function $f(x)$ is given. At the nth stage of approximation, risk of hit occurs at the n points $x = a + k(b-a)/n$ $(k = 1, 2, \cdots, n)$. The probabilities of hit are

$$p_{n,k} = \frac{1}{n} f(a + k(b - a)/n).$$

We have (setting $x_k = a + k(b-a)/n$ and $\Delta x = (b-a)/n$)

$$m(n) = \frac{1}{n} \sum_{k=1}^{n} f(a + k(b - a)/n)$$

$$= \frac{1}{b - a} \sum_{k=1}^{n} f(x_k)\Delta x \to \frac{1}{b - a} \int_a^b f(x)dx = m;$$

and (since $f(x)$ is bounded on $a \leqq x \leqq b$)

$$\max_{1 \leqq k \leqq n} p_{n,k} \leqq \frac{M}{n} \to 0.$$

Thus the Poisson distribution of mean m is obtained. But the probability ratios vary with n. They will, however, approach limiting values (cf. (5.1)). For simplicity, suppose that $f(x) > 0$ $(a \leqq x \leqq b)$.

Then

$$\frac{p(n, k+1)}{p(n, k)} = \frac{f(x_k + \Delta x)}{f(x_k)} = 1 + \frac{f(x_k + \Delta x) - f(x_k)}{f(x_k)}$$

and, by the uniform continuity of $f(x)$ on the closed interval $a \leqq x \leqq b$, we have $f(x_k+\Delta x) - f(x_k) \to 0$ and $n \to \infty$. And since our conditions insure that $f(x)$ is bounded away from zero, $p(n, k+1)/p(n, k) \to 1$: The limiting ratios are all unity.

This example leads one to seek a broadening of the proportionality conditions, by requiring that (4.1) apply only in a limiting sense. A formulation appropriate to probability is the following:

Let (4.1) be replaced by the requirement that any array of probabilities $p'_{n,k}$ shall exist which itself satisfies (4.1), and is related to the original $p_{n,k}$ by the equations

$$\lim_{n\to\infty} \max_{1\leqq k\leqq n} \left| p_{n,k} - p'_{n,k} \right| = 0, \qquad \lim_{n\to\infty} \left[m(n) - m'(n) \right] = 0,$$

where $m'(n)$ is the expected value for the accented set. Then it is seen at once that if $p_{n,k}$ satisfy (2.2), (2.3), so do $p'_{n,k}$, and with the same m; the latter thus lead to the same Poisson distribution as the former. And, of course, Theorems IV, V apply to $a_1 + a_2 + \cdots$.

In Example 3, we are thus lead to setting $a_1 = a_2 = \cdots = 1$ and $p'_{n,k} = m/n$, where m is still the mean of $f(x)$ over (a, b). The function $f_1(x) = m$ in $(a \leqq x \leqq b)$, equal to 0 outside, is related to $p'_{n,k}$ in just the same way as the original $f(x)$ was related to $p_{n,k}$; $f_1(x)$ may be described as a sort of "effective risk function" replacing $f(x)$.

Similar considerations in the case where $f(x)$ is zero in certain subintervals of (a, b) lead to an $\{a_n\}$ sequence of zeros and ones, and to an effective risk function $f_1(x)$ equal to m when $f(x) > 0$ and 0 when $f(x) = 0$.

It may be remarked that in Examples 2, 3, the x-axis may be the time axis and the danger points, epochs.

COLUMBIA UNIVERSITY

A NOTE OF CORRECTION

CHUAN-CHIH HSIUNG

Professor L. A. Santaló has called my attention to errors in the metric characterizations (12) and (20) of my paper entitled *Invariants of intersection of certain pairs of space curves*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 623–628. The correction of these will be stated as follows.

1. In the first place, we consider the two curves C, $\overline{C}$ given by the expansions (1), (2). Through the origin 0 draw the X'-axis in the XZ-plane perpendicular to the Z-axis, and then draw the Y'-axis perpendicular to the $X'Z$-plane. Let θ, ϕ, ψ be respectively the angles between the X-, X'-axes, the Y-, Z-axes, and the osculating planes $\tau, \overline{\tau}$. If x', y', z' are the coordinates of a point in space referred to the new orthogonal coordinate system $0=X'Y'Z$, then the expansions of the curves C, $\overline{C}$ in the neighborhood of the point 0 become

$$C\colon y' = -r\tan^2\psi x'^3 + \cdots, \quad z' = \tan\theta x' + a\tan^2\psi x'^2 + \cdots;$$

$$\overline{C}\colon x' = \cot\psi y' - \rho\cot\psi y'^3 + \cdots, \quad z' = \csc\psi\cot\phi y' + \alpha y'^2 + \cdots.$$

Thus we can easily obtain the following metric characterization, instead of (12), of the invariant I given by equation (6):

$$I = -\frac{\cos^3\theta}{\sin^3\phi}\frac{R\overline{T}}{T\overline{R}}.$$

2. We next consider the two curves C, $\overline{C}$ given by the expansions (15), (16). Without loss of generality we may assume the Z-axis to be perpendicular to the Y-axis. Through the origin 0 draw the X'-axis perpendicular to the YZ-plane. Let ω, θ be respectively the angles between the X-, Y-axes, and the osculating planes $\tau, \overline{\tau}$. If x', y', z' are the coordinates of a point in space referred to the new orthogonal coordinate system $0=X'YZ$, then the expansions of the curves C, $\overline{C}$ in the neighborhood of the point 0 become

$$C\colon z' = \frac{1}{\tan\theta}(x' - rx'^3) + \cdots,$$

$$y' = \frac{1}{\sin\theta\tan\omega}(x' - ax'^2) + \cdots;$$

$$\overline{C}\colon x' = -\rho\sin^3\theta\tan^3\omega y'^3 + \cdots,$$

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$$z' = -\alpha \sin\theta \cos\theta \tan^2 \omega y'^2 + \cdots .$$

Thus we can easily obtain the following metric characterization, instead of (20), of the invariant J given by equation (19):

$$J = \left(\frac{2 \sin \omega}{3 \sin \theta}\right)^6 \frac{R^2 \overline{R}^4}{T^6 \overline{T}} .$$

THE UNIVERSITY OF WISCONSIN

A REMARK ABOUT OUR NOTE "TRANSCENDENCE OF FACTORIAL SERIES WITH PERIODIC COEFFICIENTS"[1]

VERNE E. DIETRICH AND ARTHUR ROSENTHAL

Dr. B. McMillan has called our attention to his paper *A note on transcendental numbers*, Journal of Mathematics and Physics vol. 18 (1939) pp. 28–33. Moreover, Prof. C. D. Olds has referred us to the article *On transcendental numbers* by T. Itihara and K. Ôishi, Tôhoku Math. J. vol. 37 (1933) pp. 209–221. We had not known both these papers, which are closely related to our result. In the meantime Prof. J. Popken, in his review of our note (Mathematical Reviews vol. 11 (1950) p. 331), also called attention to B. McMillan's general theorem II, but remarked: "However, McMillan's proof is not quite correct and his results need supplementing." (Concerning this see a forthcoming paper of Prof. J. Popken.) On the other hand our result does not formally follow from T. Itihara's Theorem 1 (which admits k exceptional values). But our result can indeed easily be obtained also by the method applied in the proofs of T. Itihara's Theorem 1 or B. McMillan's Theorem II. However, our method is different and more direct.

PURDUE UNIVERSITY

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[1] Bull. Amer. Math. Soc. vol. 55 (1949) pp. 954–956.

ERRATA, VOLUME 1

A. M. Gleason, *Spaces with a compact Lie group of transformations.*

p. 37, line 3 from the bottom. For "C is a neighborhood of p" read "V is a neighborhood of p."

P. W. Carruth, *Roots and factors of ordinals.*

p. 471, line 21. For "$s<0$" read "$s>0$."

p. 471, line 23. For "$p_j>c_j$" read "$p_j<c_j$."

Benjamin Lepson, *Certain best possible results in the theory of Schnirelmann density.*

p. 593, line 4. For "greater" read "less."

p. 593, line 7 from the bottom. For "$\leqq$" read "$<$."

p. 593, line 5 from the bottom. For "$\leqq$" read "$<$."

p. 593, line 3 from the bottom. For

$$\text{"}\lim_{n\to\infty}\text{"}$$

read

$$\text{"}\limsup_{n\to\infty}\text{."}$$

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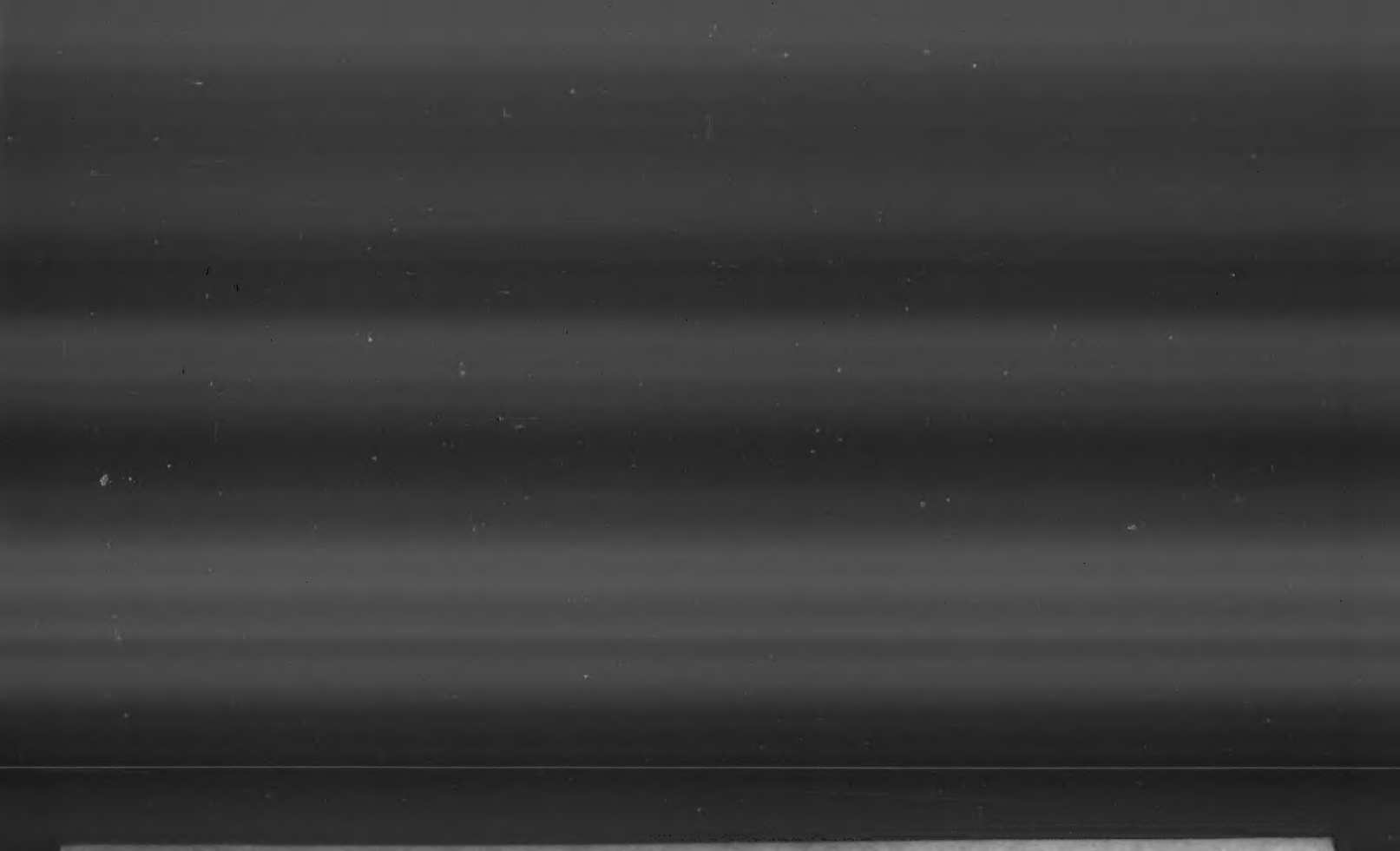
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